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TWO DIMENSIONAL SINK FLOW OF A VISCOUS, HEAT-CONDUCTING COMPRESSIBLE FLUID; CYLINDRICAL SHOCK WAVES

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Summary

The steady two-dimensional sink-type flow of a viscous, heat-conducting, perfect gas is investigated. An approximate solution of this problem is given for the case of large Reynolds number Re (cf. the definition given in the text). In obtaining the present solution the values of Prandtl number and the ratio of the first and second viscosity coefficient may be arbitrary. The result shows that the solution has two branches, both of physical significance. On the subsonic branch of the solution the flow speed starts from the stagnation point at infinity, increases monotonically for decreasing radial distance and eventually terminates with maximum speed at a certain point inside the inviscid sonic circle. The solutions of the supersonic branch, which start with the maximum speed at infinity, all contain cylindrical shocks. Within the shock the flow speed assumes a minimum value and after the shock all solutions tend asymptotically to the subsonic branch. In contrast to the plane shock, the cylindrical shock strength is limited to the order $O(Re^{-1/3})$, and the shock-thickness, of $O(Re^{-2/3})$. The latter quantity implies that the thickness of the region in which the viscous effects are important is thinner, in order of magnitude, than that of ordinary boundary layer (of $O(Re^{-1/2})$), but is thicker than that of plane normal shock (of $O(Re^{-1})$). It is found that the entropy of the supersonic branch rises to a maximum within the shock while for the subsonic branch, the entropy increases monotonically with the radial distance. The total variation of the entropy across the shock is found to be of $O(Re^{-2/3})$, which is in general greater than that across a plane normal shock $\left(\sim O(\text{shock strength})^3\right)$. The effect on the flow quantities due to the variation in viscosity coefficients, assumed to depend on the local temperature, is found to be at most of $O(Re^{-2/3})$.

Introduction

The problem of the steady cylindrical source-type or sink-type flow has been of interest to fluid-dynamicists for several reasons. First, it is known that the corresponding problem of an inviscid compressible fluid has an exact solution containing a limit line of rather special type, namely, the sonic circle (Ref. 1). To the exterior of this circle the solution has two branch of values, one has its stagnation point at infinity (subsonic branch) and the other starts with maximum velocity at infinity (supersonic branch). Both of these two branches terminate at the limit line with infinite velocity gradient. Therefore the viscous and heat-conductive effects are expected to play an important role in continuing the solution further inward. Second, because of its cylindrical symmetry, this problem is one of the few nonlinear flows in more than one dimension for which there is only one independent variable, the radial distance. Consequently, the equations are simple enough to allow a unified discussion of the various effects. These are perhaps the reasons why this problem has attracted the attention of several authors (Ref. 2, 3, 4).

In the first part of this investigation the Navier-Stokes equations are given for the cylindrical sink flow of a viscous heat-conducting perfect gas. The energy equation is integrated once to give a first order differential equation. Then, with some simplifying assumptions, the qualitative properties of the solutions are discussed in detail for the case of large Reynolds number Re. The definition of Re is $Re = \frac{\rho_1 a_1 r_1}{\mu_1}$, where $r = r_1$ locates the inviscid sonic circle with sonic speed a_1 and ρ_1, μ_1 are the fluid density and viscosity coefficient at $r = r_1$. These basic properties of the solutions thus comprehended serve for a useful guide in our final calculation of the solution.

In the second part of this paper the detailed calculation of the solution is carried out for the case of large Re. It is found that there is no single expression available for the solution uniformly valid in the entire flow region. The calculation is then performed in three different regions characterized by the length r_1 and the parameter Re. For $r > r_1 + O(Re^{-2/3})$ the approximate solution is obtained by using the PLK method*. The result fails to be a good approximation for r too close to r_1. For $r_1 - O(Re^{-2/3}) < r < r_1 + O(Re^{-2/3})$, a different similarity rule for the variables leads to a system of cylindrical transonic equations which governs the flow across the sonic region. These equations can be integrated analytically for each of the different order terms. The result shows that the solutions belonging to the supersonic branch all contain cylindrical shock-type flow in this transonic region. In other words, these solutions gradually deviate away from the inviscid supersonic branch, reach a minimum near $r = r_1$ and then approach asymptotically to the viscous subsonic branch. It is also found that the shock strength is of $O(Re^{-1/3})$ while the shock-thickness, of $O(Re^{-2/3})$, results which are quite different from that of the plane normal shock. Within this region, the thermodynamic variables satisfy the isentropic relation up to the order $O(Re^{-1/3})$ but deviate from it by a

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* This terminology was introduced by Prof. H. S. Tsien in a series of seminars, held in California Institute of Technology in 1954, in which the method due to Poincaré — Lighfill — Kuo was discussed.

quantity of $O(Re^{-2/3})$. The approximate solution for $r < r_1 - O(Re^{-2/3})$ is subsequently carried out. Finally, the entropy variation of the fluid and the effect due to variation in viscosity coefficients are discussed.

The corresponding source flow problem was previously solved, using numerical method, by Sakurai (Ref. 3); a qualitative investigation on this problem was later elaborated on many points by Levey (Ref. 4), by making use of some conventional methods in nonlinear differential equations. In the latter work, the orders of magnitude of many flow quantities of interest were estimated. The present investigation on the sink flow is not merely a special case of the cylindrical flow other than the source type, but also presents an improved method superior to those used in the previous works (e.g. Ref. 2, 3, 4). The powerful PLK-method applied to the outer region yields a set of reliable boundary values for the transonic region and thus enables all flow quantities of interest to be calculated quantitatively in all regions. However, the previous works should be credited here as to have led this author to a better understanding of this problem. The author is also indebted to Prof. H. S. Tsien for suggesting the problem and to Profs. M. S. Plesset and C. R. DePrima for their assistance on many points.

1. The Fundamental Equations

Here we are concerned with the two-dimensional sink flow of a viscous, compressible, heat-conducting fluid with polar symmetry. The only independent variable is the radial distance r from the origin. The radial velocity, u, is the only velocity component and is always nonpositive for sink flow. Let p, ρ, T, μ, μ', λ, R, c_v, c_p denote respectively the pressure, density, absolute temperature, coefficients of shear and bulk viscosity, heat conductivity, gas constant, specific heats at constant volume and pressure. The momentum equation is

$$\rho u \frac{du}{dr} = -\frac{dp}{dr} + \frac{d}{dr}\left[2\mu \frac{du}{dr} + \frac{2}{3}(\mu'-\mu)\frac{1}{r}\frac{d}{dr}(ru)\right] + 2\mu \frac{d}{dr}\left(\frac{u}{r}\right) \tag{1.1}$$

and the energy equation is

$$\rho u r \frac{d}{dr}\left(\frac{u^2}{2} + c_p T\right) = \frac{d}{dr}\left\{r\left[\lambda \frac{dT}{dr} + \mu \frac{du^3}{dr} + \frac{2}{3}(\mu'-\mu)\left(\frac{1}{2}\frac{du^2}{dr} + \frac{u^2}{r}\right)\right]\right\} \quad . \tag{1.2}$$

The continuity equation can be written in the following form if m denotes the sink strength,

$$2\pi \rho u r = -m \tag{1.3}$$

The equation of state is assumed to be that of a perfect gas,

$$p = R\rho T \quad . \tag{1.4}$$

Equation (1.1-1.4) are a system of nonlinear differential equations for four variables u, p, ρ and T if μ, μ', λ, and c_p are known functions of T.

To reduce the equations to nondimensional form, the following nondimensional quantities are introduced;

$$\left.\begin{aligned} &\bar{r} = (r/r_1) \quad , \quad \eta = \log \bar{r} \quad , \quad w = -u/a_1 \quad , \quad \theta = T/T_1 = (a/a_1)^2 , \\ &\bar{p} = p/p_1 \quad , \quad \bar{\rho} = \rho/\rho_1 \quad , \quad \bar{\mu} = \mu/\mu_1 \quad , \quad \bar{\mu}' = \mu'/\mu_1 \end{aligned}\right\} \tag{1.5}$$

where quanities with the subscript 1 are ficticious quanties which would occur at the local Mach number unity for nonviscous and non-heat-conducting gas. Thus, with γ equal to the ratio of specific heats, assumed constant throughout, the sonic speed a_1 at $r=r_1$ is given by

$$a_1^2 = \gamma p_1/\varrho_1 \quad , \text{ and } \quad 2\pi \varrho_1 a_1 r_1 = m \quad . \tag{1.6}$$

The continuity equation then becomes

$$\bar{\varrho}\, w \bar{r} = 1 \quad . \tag{1.7}$$

Here w is always positive for sink flow. The equation of state is now

$$\bar{p} = \bar{\varrho}\, \theta \quad . \tag{1.8}$$

Using Eq. (1.7), and writing

$$\mu' - \mu = 3 k \mu \quad , \tag{1.9}$$

the nondimensional form of Eq. (1.1) becomes

$$\frac{1}{\bar{r}} \frac{dw}{d\bar{r}} = -\frac{1}{\gamma} \frac{d\bar{p}}{d\bar{r}} - 2\bar{\alpha} \left\{ \frac{d}{d\bar{r}} \left[\bar{\mu} \frac{dw}{d\bar{r}} + k\bar{\mu} \frac{1}{\bar{r}} \frac{d}{d\bar{r}} (\bar{r} w) \right] + \bar{\mu} \frac{d}{d\bar{r}} \left(\frac{w}{\bar{r}} \right) \right\} \tag{1.10}$$

where $\bar{\alpha}$ denotes the inverse of the Reynolds number

$$\bar{\alpha} = \frac{\mu_1}{\varrho_1 a_1 r_1} = \frac{2\pi \mu_1}{m} \quad , \tag{1.11}$$

which will be considered much smaller than unity throughout this paper. Stokes assumption on the viscosity coefficients states that

$$\mu' = 0 \quad , \text{ or } \quad k = -\frac{1}{3} \quad . \tag{1.12}$$

As this assumption does not agree with observations for many kinds of fluid (cf. Ref. 5), condition (1.12) will not be imposed on the final calculation of the flow field.

Again using Eq. (1.7), the nondimensional form of Eq. (1.2) can be integrated once to yield:

$$\frac{w^2}{2} + \frac{\theta}{\gamma-1} + \bar{\alpha}\bar{r}\bar{\mu} \left[\frac{1}{\sigma} \frac{d}{d\bar{r}} \left(\frac{\theta}{\gamma-1} \right) + (1+k) \frac{dw^2}{d\bar{r}} + 2k \frac{w^2}{\bar{r}} \right] = \frac{\gamma+1}{2(\gamma-1)} \tag{1.13}$$

where σ denotes the Prandtl number

$$\sigma = \frac{c_p \mu}{\lambda} \quad . \tag{1.14}$$

The integration constant on the right hand side of Eq. (1.13) is chosen as shown above so that the limit solution for vanishing viscosity agrees at large r with that of a nonviscous, iso-energetic flow.

Eliminating $\bar{p}$ in Eq. (1.10) by using (1.7) and (1.8), and introducing $\eta = \log \bar{r}$ as the independent variable, we obtain:

$$\frac{dw}{d\eta} + \frac{1}{\gamma}\left[\frac{d}{d\eta}\left(\frac{\theta}{w}\right) - \frac{\theta}{w}\right] = -2\bar{\alpha}\left\{\bar{\mu}(1+k)\left(\frac{d^2w}{d\eta^2} - w\right) + \left[(1+k)\frac{dw}{d\eta} + kw\right]\frac{d\bar{\mu}}{d\eta}\right\} \quad . \tag{1.15}$$

The energy equation in terms of η is now

$$\frac{w^2}{2} + \frac{\theta}{\gamma-1} + \bar{\alpha}\bar{\mu}\left[\frac{\sigma^{-1}}{\gamma-1}\frac{d\theta}{d\eta} + (1+k)\frac{dw^2}{d\eta} + 2kw^2\right] = \frac{\gamma+1}{2(\gamma-1)} \quad . \tag{1.16}$$

Eqs. (1.15) and (1.16) are the two equations for two unknowns w and θ. The boundary conditions for them can be determined by requiring that they tend to their respective inviscid solutions as $\eta \to \infty$ so that these two solutions can be appropriately compared later.

1.1 The Nonviscous Solution

The solution for sink flow of a compressible inviscid gas can be literally obtained by putting $\bar{\alpha} = 0$ in the above equations without justifying the validity of such a simplification. Then the equations reduce to

$$\frac{dw}{d\eta} + \frac{1}{\gamma}\left[\frac{d}{d\eta}\left(\frac{\theta}{w}\right) - \frac{\theta}{w}\right] = 0 \tag{1.17}$$

and

$$\theta + \frac{\gamma-1}{2}w^2 = \frac{\gamma+1}{2} \quad . \tag{1.18}$$

The solution of this system of equations is known to be

$$\frac{r}{r_1} = e^{\eta} = w^{-1}\left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2} w^2\right)^{-\frac{1}{\gamma-1}} , \qquad (1.19)$$

and

$$\theta = \left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2} w^2\right) , \quad \bar{p} = \bar{\rho}^{\gamma} = \theta^{\frac{\gamma}{\gamma-1}} . \qquad (1.20)$$

The value of r_1 can be expressed in terms of stagnation state as

$$r_1 = \frac{-m}{2\pi \rho_1 u_1} = \frac{m}{2\pi a_1 \rho_1} = \frac{m}{2\pi a_0 \rho_0}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (1.21)$$

where $a_0^2 = \frac{\gamma p_0}{\rho_0}$, and p_0 , ρ_0 are the isentropic stagnation pressure and density. Equation (1.20) simply states the isoenergetic and isentropic relations.

This inviscid solution $w(r)$ given by Eq. (1.19) is plotted in Fig. 1.

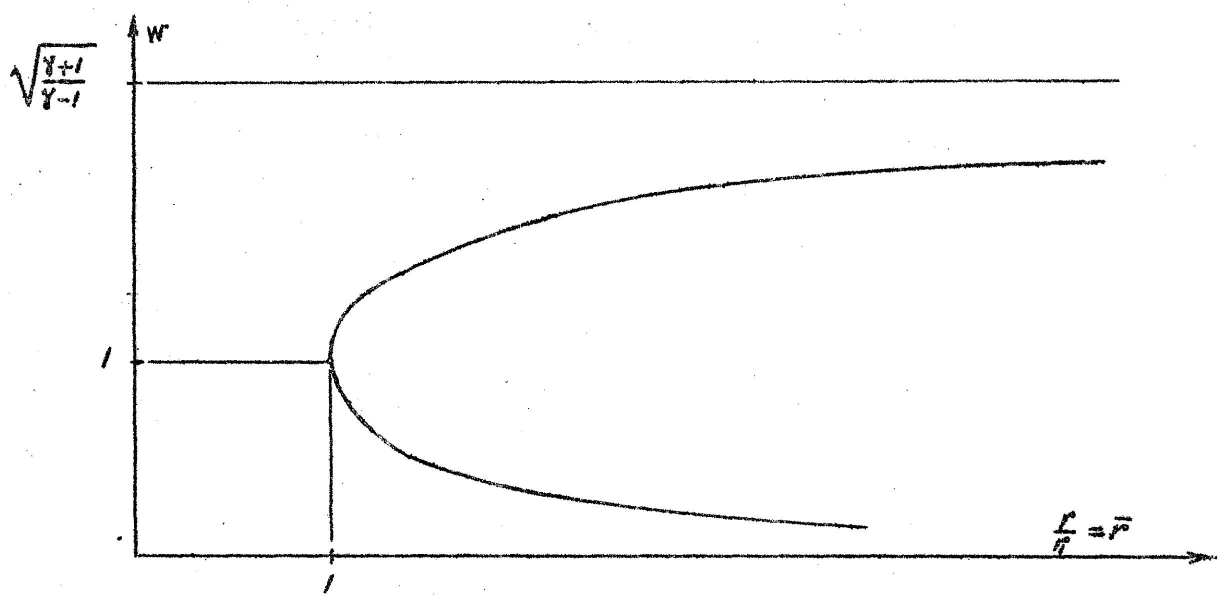

Fig. 1. Graph of Inviscid Solution.

It gives no solution for $r < r_1$, but for $r > r_1$, w is a double-valued function of r . On one branch w tends to zero so that thermodynamic variables tend to their stagnation values as $r \to \infty$; on the other branch w tends to maximum speed attainable $\sqrt{\frac{\gamma+1}{\gamma-1}}$, and the thermodynamic variables tend to zero as $r \to \infty$. They will be designated as subsonic and supersonic branch respectively. Both of the branches terminate at $r = r_1$ with sonic speed (at which the fluid speed equals the local speed of sound). The slope of the curve $w(r)$,

$$\frac{dw}{d\bar{r}} = \frac{2}{\gamma+1} \; \frac{w^3 \left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2} w^2 \right)^{\frac{\gamma}{\gamma-1}}}{w^2 - 1} \quad , \tag{1.22}$$

is much smaller than unity for $r \gg r_1$ on both branches and consequently viscous effects become comparatively unimportant there. But as $r \to r_1$, $w \to 1$, $\frac{dw}{dr}$ becomes numerically unbounded, and thus the viscous force near $r = r_1$ should play a role as significant as those of inertia and pressure forces.

Now this inviscid solution will be used as a guide to study the sink flow of a real fluid governed by Eqs. (1.15, 1.16) for large values of Re, in the sense that it is assumed that the limit of the viscous solution for vanishing viscosity approaches the inviscid solution as $r \to \infty$, for both subsonic and supersonic branches. By continuing these viscous solutions backward in r where viscous effects become more and more prominant, it is expected that the real fluid, affected by viscosity and heat conduction, will flow across this fictitious sonic circle, which is a limit line when viscosity is neglected.

It may be remarked here that the equations for source flow of real fluid can be obtained from (1.15) and (1.16) by changing the sign of terms with factor $\bar{\alpha}$ if w again represents the absolute value of the radial speed, normalized relative to q_1 (cf. Ref. 4). Hence the inviscid solutions for source and sink flow are identical, but their respective viscous solutions will be shown later to have quite different features for all r .

2. Properties of the Solution Curves

2.1 Approximate Differential Equation in the "Phase Space".

In order to study the qualitative properties of the solution curves, several assumptions will be introduced in this section to simplify the analysis while most of the important features of the original system will still be maintained. The Prandtl number, σ , is assumed constant because μ and λ have almost the same dependence on temperature. In this section, μ is also taken to be constant so that $\bar{\mu} = 1$. When the complete solution is calculated later, however, the assumptions introduced here become unnecessary.

Equation (1.13) can be integrated when the Prandtl number

$$\sigma = \frac{1+4\bar{\alpha}k}{2(1+k)} , \qquad (2.1)$$

(under Stokes assumption, $k = -\frac{1}{3}$, then $\sigma = \frac{3}{4} - \bar{\alpha}$), and the final integral is

$$\theta - \left[\frac{\gamma+1}{2} - \frac{\gamma-1}{2}(1+4\bar{\alpha}k)w^2\right] = \bar{A}\, e^{-\frac{\sigma}{\bar{\alpha}}\eta} = \bar{A}\left(\frac{r}{r_1}\right)^{-\frac{\sigma}{\bar{\alpha}}} , \qquad (2.2)$$

where $\bar{A}$ is the integration constant. The value chosen above for σ is actually not far from experimental data ($\sigma \doteq 0.72$ for air at standard condition). As σ only appears in the coefficient of the derivative $\frac{d\theta}{d\eta}$ in Eq. (1.16), it follows from the theory of differential equations (Ref. 6 p. 142) that the solutions and all their derivatives will be continuous in σ for $w > 0$, $-\infty < \sigma < \infty$. Thus the assumption of choosing this particular value of σ would merely lead to simplification of analysis rather than material change of the solutions. If we further require by physical argument that the deviation from the iso-energetic relation expressed by the term with the arbitrary constant $\bar{A}$ will not overwhelm the right hand side terms for $r<r_1$, we may assume that $\bar{A}=0$. This restriction, however, can again be relaxed when the complete solution is discussed later. It will then be shown that $\bar{A}$ is indeed of the order $O(\alpha)$. Thus the particular solution with $\bar{A}=0$ would still provide a good approximation to the complete solution.

With $\bar{A}=0$, the energy relation becomes

$$\theta = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}(1+4\bar{\alpha}k)w^2 \quad . \tag{2.3}$$

Introducing this equation into Eq. (1.15) and eliminating the explicit dependence on η by the substitution

$$V = -\frac{dw}{d\eta} = -r\frac{dw}{dr} \quad , \tag{2.4}$$

we obtain

$$\alpha w^2 V \frac{dV}{dw} + V\left[1-(1-a\alpha)w^2\right] - w\left\{1-\left[\beta+(a-1)\alpha\right]w^2\right\} = 0 \tag{2.5}$$

where

$$\alpha = \frac{4\gamma}{\gamma+1}(1+k)\bar{\alpha} \quad , \quad \beta = \frac{\gamma-1}{\gamma+1} \quad , \quad a = \frac{k}{1+k}\left(\frac{\gamma-1}{\gamma}\right) \quad . \tag{2.5a}$$

The variable V is closely related to the fluid velocity gradient. Since the terms $a\alpha$ and $(a-1)\alpha$ in the brackets are merely corrections to constant coefficients of $O(1)$, the properties of Eq. (2.5) would not be altered if we had neglected these terms in order to simplify further algebra. Thus the approximate differential equation

$$\alpha w^2 V \frac{dV}{dw} + V(1-w^2) - w(1-\beta w^2) = 0 \tag{2.6}$$

in the phase space (w, V) is expected to exhibit all important features of the original system Eqs. (1.15 - 16) for $\bar{\mu}=1$. The equation similar to (2.6) was derived by Sakurai (Ref. 3) and later was discussed in detail by Levey (Ref. 4) for source flow in a real fluid.

2.2 Properties of the Solution Curves in Phase Space.

Equation (2.6) is nonlinear and cannot be integrated. However, several important features of the solutions can be readily seen by studying the properties of the vector field (w, V) defined by Eq. (2.6), such as the type of its singular points, the curves of zero slope and zero curvature together with some obvious isoclines.

(a) The curve of zero slope; the inviscid solution.

Let C_1 be the curve on which $\frac{dV}{dw}$ given in Eq. (2.6) vanishes, C_1 is then given by

$$V = \frac{w(1-\beta w^2)}{1-w^2}, \tag{2.7}$$

which is also the inviscid solution in w-V plane. The function $V(w)$ given by Eq. (2.7) has a simple pole at $w=1$ (the fictitious sonic circle), and two zeros at $w=0$ and $w=\beta^{-\frac{1}{2}}$ which correspond respectively to the subsonic and supersonic branch at $r=\infty$. Near the origin, $V(w)$ has the following power series expansion

$$V_{C_1} = w\left[1+(1-\beta)w^2+(1-\beta)w^4+(1-\beta)w^6+\cdots\right] \tag{2.8}$$

which starts from $w=0$ with slope unity. Near the point $w=\beta^{-\frac{1}{2}}$, the expansion of $V(w)$ is

$$V_{C_1} = \frac{2\beta x}{1-\beta}\left[1-\frac{1+3\beta}{2(1-\beta)}(\sqrt{\beta}\,x)+\frac{1+6\beta+\beta^2}{2(1-\beta)^2}(\sqrt{\beta}\,x)^2-\frac{1+10\beta+5\beta^2}{2(1-\beta)^3}(\sqrt{\beta}\,x)^3+\cdots\right] \tag{2.9}$$

where

$$x = w - \beta^{-\frac{1}{2}} .$$

This branch of the curve C_1 starts from $(\beta^{-\frac{1}{2}}, 0)$ with the slope $\frac{2\beta}{1-\beta}$. The curve C_1 divides the infinite strip $0 \leq w \leq \beta^{-\frac{1}{2}}$ into regions of positive and negative slope as shown in Fig. 2.

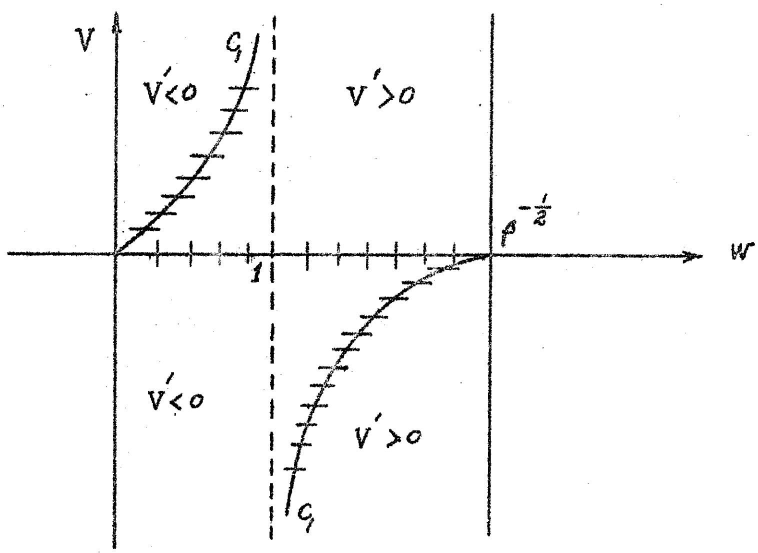

Fig. 2. Regions of Positive and Negative Slope; Isoclines .

In this strip, $\frac{dV}{dw}$ given in Eq. (2.6) becomes infinite only on (i) $V=0$, $0 < w < \beta^{-\frac{1}{2}}$, and (ii) $w=0$, $V \neq 0$. Besides, $w = \beta^{-\frac{1}{2}}$, $V \neq 0$ is also an isocline on which

$$\frac{dV}{dw} = \frac{1-\beta}{\alpha} .$$

(b) Properties of the singular points.

The only singular points of Eq. (2.6) are $(0,0)$ and $(\beta^{-\frac{1}{2}}, 0)$. The origin is a singular point of higher order. But if one sketches the vector field $({dw}/{dt} , {dV}/{dt})$ defined by

$$\frac{dw}{dt} = \alpha w^2 V \quad , \quad \frac{dV}{dt} = w(1-\beta w^2) - V(1-w^2)$$

along a simple closed curve $C(t)$ in the neighborhood of the origin with the origin in its interior, one finds that the Poincaré index (cf. Ref. 7 p. 45) of this singularity is equal to -1. Thus the origin is a saddle point through which only two solution curves may pass. One of these is $w=0$ which either yields a trivial solution $(V=0)$ or has no physical meaning $(V\neq 0)$. The other solution curve starts from the origin with slope equal to $+1$ (which coincides there with the inviscid solution) and thus represents the only possible radial sink flow with stagnation at $r=\infty$. By substitution of a power series into Eq. (2.6) (or by ordinary iteration), the asymptotic value, for small α, of this solution near the origin is found to be

$$V \cong w + \left[(1-\beta)-\alpha\right]w^3 + \left[(1-\beta)-\alpha(5-4\beta)+4\alpha^2\right]w^5 + \left[(1-\beta)-\alpha(14-16\beta+3\beta^2)+10\alpha^2(4-3\beta)-27\alpha^3\right]w^7 + \cdots \tag{2.10}$$

The point $(\beta^{-\frac{1}{2}}, 0)$ is a singular point of regular type. In the neighborhood of $x \equiv w-\beta^{-\frac{1}{2}}=0$, $V=0$, Eq. (2.6) becomes

$$\frac{dV}{dx} = \frac{-2\beta x+(1-\beta)V+P(x,y)}{\alpha V+Q(x,y)} \tag{2.11}$$

in which P and Q vanish like x^2+V^2 as $x, V \to 0$. Thus the singularity (Ref. 7 pp. 37-44) is

$$\begin{aligned} &\text{(i) a nodal point if } \alpha \leq \frac{(1-\beta)^2}{8\beta}, \quad \text{(for air, } \beta=\tfrac{1}{6}, \ \alpha \leq \tfrac{35}{48}\text{)} \\ &\text{(ii) a spiral point if } \alpha > \frac{(1-\beta)^2}{8\beta} \end{aligned} \tag{2.12}$$

As the problem will be confined to the case $\alpha \ll 1$, we shall only consider this singularity to be a nodal point, (which changes to a saddle point for source flow, cf. Ref. 4). All solution curves passing through this point will have at this point two distinct slopes which can be calculated from the secular equation of Eq. (2.11), namely,

$$\begin{vmatrix} \lambda & -\alpha \\ 2\beta & \lambda-(1-\beta) \end{vmatrix} = 0 \quad , \tag{2.13}$$

which has two unequal positive roots

$$\lambda_1 = (1-\beta) - \frac{2\alpha\beta}{(1-\beta)} + O(\alpha^2) \quad , \quad \lambda_2 = \frac{2\alpha\beta}{1-\beta} + O(\alpha^2) \quad . \tag{2.13a}$$

From these two eigen-values two eigen-vectors associated with Eq. (2.11) at $(x=0, V=0)$ can be obtained. By using this result it can be shown that the solution curves passing through $(x=0, V=0)$ have near this point the following parametric representation

$$\left.\begin{aligned} V - \frac{2\beta}{1-\beta}\, x &= c_1 e^{\lambda_1 t} \\ V - \left(\frac{1-\beta}{\alpha} - \frac{2\beta}{1-\beta}\right) x &= c_2 e^{\lambda_2 t} \end{aligned}\right\} \tag{2.14a}$$

with the slope

$$\frac{dV}{dx} = \frac{dV/dt}{dx/dt} = \frac{\frac{2\beta}{1-\beta} c_2 \lambda_2 - \left(\frac{1-\beta}{\alpha} - \frac{2\beta}{1-\beta}\right) c_1 \lambda_1 e^{(\lambda_1-\lambda_2)t}}{c_2\lambda_2 - c_1\lambda_1 e^{(\lambda_1-\lambda_2)t}} \tag{2.14b}$$

Since $\lambda_1 > \lambda_2 > 0$, $V, x \to 0$ as $t \to -\infty$. It also follows from the above equations that there are infinite number of solution curves, corresponding to arbitrary c_1 and c_2 $(c_2 \neq 0)$ which have the asymptotic value

$$V \cong \frac{2\beta}{1-\beta}\, x + C\,|x|^{\frac{(1-\beta)^2}{2\alpha\beta}} \qquad \text{near } (x=0, V=0) \; ; \tag{2.15a}$$

and, in addition, there is another solution curve passing through this point $(\sim C_2 = 0)$, of the value

$$V \cong \left(\frac{1-\beta}{\alpha} - \frac{2\beta}{1-\beta}\right) x \qquad \text{near} \qquad (x=0,\ V \doteq 0). \tag{2.15b}$$

The first group of solutions, given by Eq. (2.15a), have the same limiting value at $r=\infty$ as the inviscid solution and hence represent the many possible sink flows starting with maximum velocity $w=\beta^{-\frac{1}{2}}$ at $r=\infty$, while the solution given by Eq. (2.15b) is physically irrelevant. The asymptotic value, for small α , of this physically significant solution near the point $w=\beta^{-\frac{1}{2}}$ is

$$\begin{aligned} V \cong \frac{2\beta x}{1-\beta}\Bigg\{ & \left[1+\frac{2\alpha\beta}{(1-\beta)^2}+\frac{8\alpha^2\beta^2}{(1-\beta)^4}+O(\alpha^3)\right] - \frac{\sqrt{\beta}\,x}{2(1-\beta)}\left[(1+3\beta)+2(3+13\beta)\frac{\alpha\beta}{(1-\beta)^2}+O(\alpha^2)\right] \\ & + \frac{\beta x^2}{2(1-\beta)^2}\left[(1+6\beta+\beta^2)+2(5+42\beta+33\beta^2)\frac{\alpha\beta}{(1-\beta)^2}+O(\alpha^2)\right]+\cdots\Bigg\} \\ & + C\,|x|^{(1-\beta)^2/2\alpha\beta} \qquad \left(\text{near} \quad x = w-\beta^{-\frac{1}{2}} = 0\right) \end{aligned} \tag{2.16}$$

where C is an arbitrary constant. The last term follows from Eq. (2.15a).

(c). The curve of zero curvature.

The second derivative of the solution V can be deduced from Eq. (2.6),

$$\alpha w^3 V^3 \frac{d^2V}{dw^2} = 2V^3 - w(1+\beta w^2)V^2 + \frac{1}{\alpha}(1-\beta w^2)\left[V(1-w^2) - w(1-\beta w^2)\right]. \tag{2.17}$$

Let C_2 be the curve on which $\frac{d^2V}{dw^2}$ vanishes, the equation for C_2 is then, except where wV is zero, given by:

$$2V^3 - w(1+\beta w^2)V^2 + \frac{1}{\alpha}(1-\beta w^2)\left[V(1-w^2) - w(1-\beta w^2)\right] = 0 \tag{2.18}$$

The function $V_{C_2}(w)$ satisfying this equation has the following properties:

(i) V_{C_2} has only one real value for either

$$0 \leq w < 1 + \frac{3}{2}\left[\frac{(1-\beta)}{2}\alpha\right]^{1/3} \qquad \text{or} \qquad w > \beta^{-\frac{1}{2}}\left[1 + \frac{\alpha}{4(1-\beta)}\right] \quad ; \tag{2.19a}$$

V_{C_2} has three real values for

$$1 + \frac{3}{2}\left[\frac{1-\beta}{2}\alpha\right]^{1/3} \leq w \leq \beta^{-\frac{1}{2}}\left[1 + \frac{\alpha}{4(1-\beta)}\right] \quad ; \tag{2.19b}$$

V_{C_2} has two equal real values at

$$w = 1 + \frac{3}{2}\left[\frac{1-\beta}{2}\alpha\right]^{1/3}, \quad w = \beta^{-\frac{1}{2}}\left[1 + \frac{\alpha}{4(1-\beta)}\right] \quad \text{and} \quad w = \beta^{-\frac{1}{2}} . \tag{2.19c}$$

(ii) The curve C_2 starts from the origin with slope equal to unity and has, in the neighborhood of the origin, the following expansion:

$$\begin{aligned} V_{C_2} \cong\ & w + \left[(1-\beta) - \alpha\right] w^3 + \left[(1-\beta) - \alpha(5-4\beta) + 4\alpha^2\right] w^5 \\ & + \left[(1-\beta) - \alpha(14 - 16\beta + 3\beta^2) + 2\alpha^2(17-12\beta) - 21\alpha^3\right] w^7 + \cdots \end{aligned} \tag{2.20}$$

(iii) The curve C_2 crosses $w = 1$ at

$$V_{C_2}(1) = (1-\beta)^{2/3}(2\alpha)^{-\frac{1}{3}}\left[1 + O(\alpha^{\frac{1}{3}})\right] \tag{2.21a}$$

with the slope

$$\frac{dV}{dw} = \frac{2}{3}(1-\beta)^{1/3}(2\alpha)^{-\frac{2}{3}}\left[1 + O(\alpha^{1/3})\right] \tag{2.21b}$$

(iv) When $w = \beta^{-\frac{1}{2}}$, $V_{C_2}^2 = 0$ and $V_{C_2} = \beta^{-\frac{1}{2}}$. Thus C_2 has a double point at $(\beta^{-\frac{1}{2}}, 0)$, where the curve has two different slopes $\frac{2\beta}{(1-\beta)}[1 + O(\alpha)]$ and $\frac{1-\beta}{\alpha}[1 + O(\alpha)]$. Near this point these two branches of the curve C_2 has the following expansions:

$$\begin{aligned} V_{C_2}^{(1)} = \frac{2\beta x}{(1-\beta)}\Bigg\{ & \left[1 + \frac{2\alpha\beta}{(1-\beta)^2} + \frac{8\alpha^2\beta^2}{(1-\beta)^4} + O(\alpha^3)\right] - \frac{\sqrt{\beta}\,x}{2(1-\beta)}\left[(1+3\beta) + 4(7\beta-1)\frac{\alpha\beta}{(1-\beta)^2} + O(\alpha^2)\right] \\ & + \frac{\beta x^2}{2(1-\beta)^2}\left[(1 + 6\beta + \beta^2) + O(\alpha)\right] x^2 + \cdots \Bigg\} \end{aligned} \tag{2.22a}$$

and

$$V_{C_2}^{(2)} = \frac{1-\beta}{\alpha} x \left\{ \left[1 - \frac{2\alpha\beta}{(1-\beta)^2} + O(\alpha^2) \right] + O(x) \right\} \tag{2.22b}$$

where $\quad x = w - \beta^{-\frac{1}{2}}$.

(v) The slope of the curve C_2

$$\left(\frac{dV}{dw}\right)_{C_2} = \frac{(1+3\beta w^2)V^2 + \frac{2}{\alpha}\left[(1+\beta) - 2\beta w^2\right] w V + \frac{1}{\alpha}(1-\beta w^2)(1-5\beta w^2)}{6V^2 - 2w(1+\beta w^2)V + \frac{1}{\alpha}(1-w^2)(1-\beta w^2)} \tag{2.23}$$

becomes infinite at

$$w \sim 1 + \frac{3}{2}\left[\frac{(1-\beta)\alpha}{2}\right]^{1/3} \quad , \quad V \sim -\left(\frac{1-\beta}{2}\right)^{2/3} \alpha^{-\frac{1}{3}}$$

and

$$w \sim \beta^{-\frac{1}{2}}\left[1 + \frac{\alpha}{4(1-\beta)}\right] \quad , \quad V \sim \frac{1}{2}\beta^{-\frac{1}{2}} .$$

(vi) For $w \gg 1$, $V_{C_1} \sim \beta w$, $V_{C_2} \sim \frac{1}{2}\beta w^3$. (2.24)

The curve C_2 divides the $(w - V)$ plane into regions of positive and negative curvatures as shown in Fig. 3.

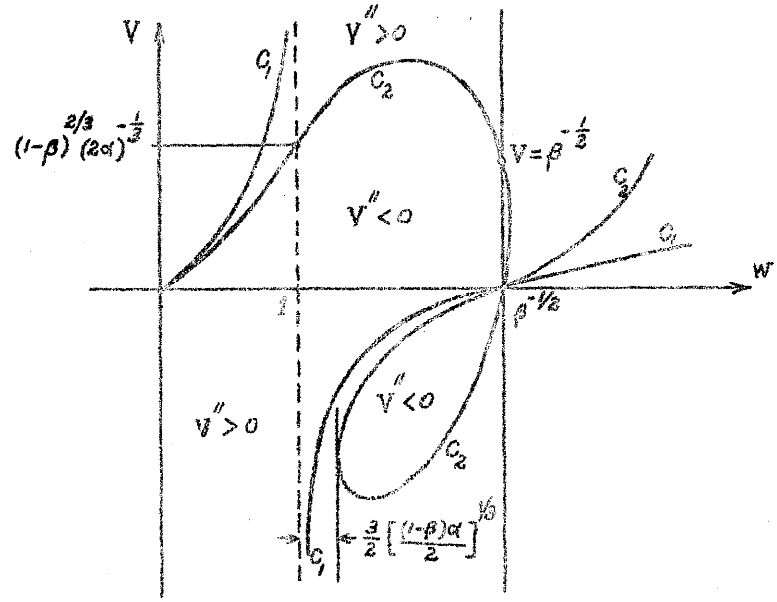

Fig. 3. Regions of Positive and Negative Curvature.

(d). Sketch of solution curves $V(w)$; definition of cylindrical shock.

The above discussions on isoclines, types of singularities, regions of positive and negative slope and curvature, enable the solution curves to be sketched.

First let us consider the solution curve starting from the origin. Comparison of Eqs. (2.8), (2.10), and (2.20) shows that, for α and w small,

$$V_{C_1}(w) > V(w) > V_{C_2}(w) \quad , \tag{2.25}$$

the difference $(V_{C_1} - V) \sim O(\alpha w^3)$ while $(V - V_{C_2}) \sim O(\alpha^2 w^7)$. Thus the solution curve $V(w)$ lies in between C_1 and C_2 where the slope and curvature of $V(w)$ are both positive. A careful study of the slope of $V(w)$ and C_2 (cf. Eqs. 2.6 and 2.23) indicates that the solution curve lies above C_2 for $w > 0$. Hence $V(w)$ is a monotonic increasing function of w with increasing slope, passing through $w = 1$ between the points $V = \frac{1-\beta}{\alpha}$ and $V = (1-\beta)^{2/3}(2\alpha)^{-\frac{1}{3}}$, and finally ending up at $w = \beta^{-\frac{1}{2}}$ with the slope $\frac{dV}{dw} = \frac{1-\beta}{\alpha}$ (cf. Fig. 4).

As previously shown in Eq. (2.16), there are infinite number of solution curves starting from $w = \beta^{-\frac{1}{2}}$, $V = 0$ with the same slope $\frac{2\beta}{1-\beta}$. However, for $(w - \beta^{-\frac{1}{2}})$ and α both small enough, comparison of Eqs. (2.9), (2.16) and (2.22a) again shows that all these solutions satisfy for small negative $(w - \beta^{-\frac{1}{2}})$, the following inequality

$$V(w) < V_{C_2}(w) < V_{C_1}(w) \tag{2.26}$$

as shown in Fig. 4.

As w decreases from $\beta^{-\frac{1}{2}}$, V (for every finite C in Eq. 2.16) decreases with increasing slope until it intersepts C_2 with a positive

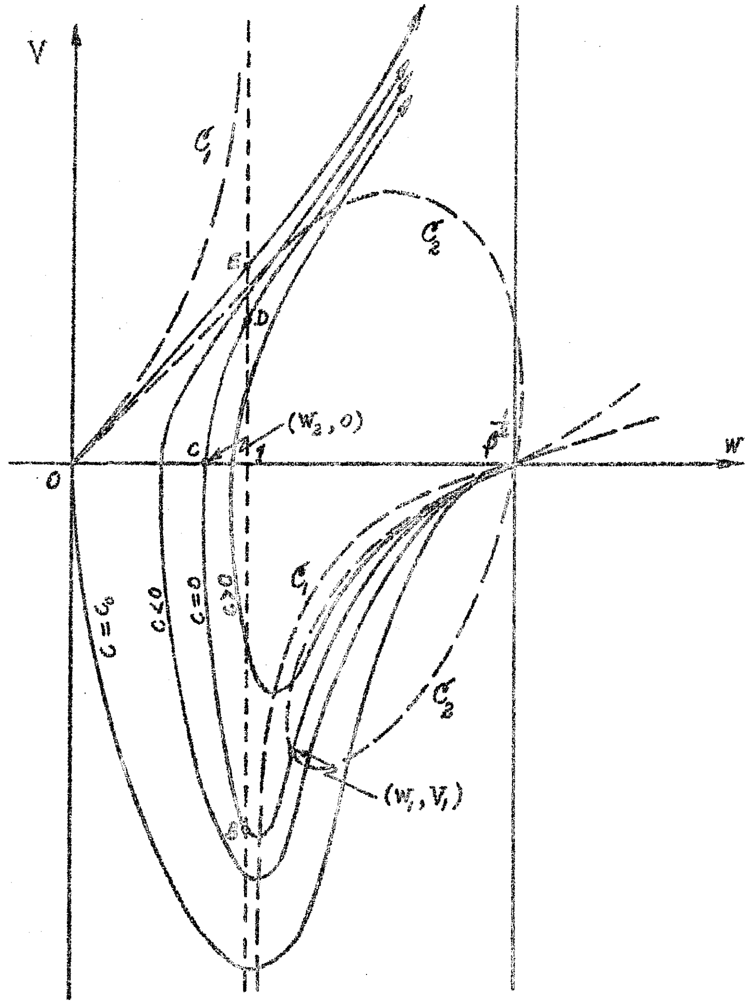

Fig. 4. Sketch of the Solution Curves in the Phase Space .

slope. For further decrease in W , the curve V should lie above a straight line with this positive slope at the point of interception because $\frac{d^2V}{dW^2}$ is positive in this region. Hence the solution curves will eventually meet $\mathcal{C}_1$ with zero slope, and from there on, for further decrease in W , V increases from negative values and later crosses $V = 0$ with infinite slope at some point in between $W = 0$ and $W = \beta^{-\frac{1}{2}}$, as can be shown by the method of bounding curves and will be made explicit in our later calculation. Further extension of these solution curves shows that V increases with increasing w and finally approaches asymptotically to the subsonic branch solution which starts from the origin. There is a particular value of the integration constant (in Eq. 2.16), say, $c = c_0 < 0$, for which the solution curve finally ends up at the origin with infinite slope. For $c < c_0$, the solution ceases to have physical meaning. On the other hand, the solution curves for $c > c_0$ have a very interesting feature that these viscous solutions all exhibit the transition process from the "inviscid supersonic branch" toward the "viscous subsonic branch". Let us consider, in particular, the solution with $c = 0$. It first intercepts $\mathcal{C}_2$ at (w_1, V_1), say, and then crosses $V = 0$ at $w = w_2$. Since $w_1 > 1$ and $w_2 < 1$ (as will be shown later), the flow between these two states may thus be defined as that of a "cylindrical shock".* Inspired by the result obtained in Eqs. (2.21), we see that the equation governing such a cylindrical shock flow can be approximated by the following similarity transformation

$$w = 1 + \alpha^{1/3}\tilde{w} \quad , \quad V = \alpha^{-\frac{1}{3}}\tilde{V} . \tag{2.27}$$

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* This terminology is adopted by both Sakurai[3] and Levey[4] to describe such type of flow. The term "shock" is borrowed from its conventional meaning to indicate the transition from one branch to the other, though the transition is rather different from that occurring in a plane normal shock. Perhaps this terminology relates closer to the conventional meaning of a shock for the constant c slightly greater than c_0 (cf. Fig. 4), because then the jump in w and the slope $\frac{dw}{d\eta}$ in transition become greater and the position of transition is farther out from $r = r_i$ (cf. Fig. 5). But since there is no adequate criterion to distinguish one from another value of C, we shall retain this name. Another terminology, the dissipation layer, is suggested by Prof. H. S. Tsien to void this ambiguity and, in addition, to stress the importance of viscous effects in this layer.

which will render all terms in Eq. (2.6) equally important in this flow region, that is, for vanishing α ,

$$\tilde{V} \frac{d\tilde{V}}{d\tilde{w}} = 2\tilde{w}\tilde{V} + (1-\beta) \quad . \tag{2.28}$$

Since this equation governs the flow of both branches near the sonic speed, it may be called the "equation for cylindrical transonic flow".

(e) Sketch of solution curves $w(\eta)$ in physical space.

From the definition of V , $V(w) = -\frac{dw}{d\eta}$, we have

$$\eta = -\int^{w} \frac{dw}{V(w)} + C \tag{2.29}$$

where the integral stands for an indefinite integral and

$$C = \frac{1}{\gamma-1} \log\left(\frac{2}{\gamma+1}\right) \tag{2.29a}$$

so that η tends to its inviscid solution for η large.

Now for the subsonic branch, $V(w) \geqslant 0$, hence η is a monotonically decreasing function of w . Moreover, for same value of w , $V(w)$ is less than its corresponding inviscid solution (cf. Eq. 2.25). Hence, from Eq. (2.29),

$$\eta_{vis.}(w) < \eta_{invis.}(w) \tag{2.30}$$

In other words, at every η , $(w)_{vis}$ is slowed down from its inviscid value due to the viscous effect.

For the supersonic branch starting from $w = \beta^{-\frac{1}{2}}$, $V(w) \leqslant 0$ for $w_2 \leqslant w \leqslant \beta^{-\frac{1}{2}}$, hence in this interval η is a monotonically increasing function of w . At $w = w_2$, ($\sim \eta = \eta_2$, say),

$$\left(\frac{dw}{d\eta}\right)_{w=w_2} = -V(w_2) = 0$$

and, from Eq. (2.6)

$$\left(\frac{d^2w}{d\eta^2}\right)_{w=w_2} = \left(V \frac{dV}{dw}\right)_{V=0} = \frac{1}{\alpha w_2}(1-\beta w_2^2) > 0 \quad ,$$

therefore $w(\eta_2) = w_2$ is the only minimum of w on this branch. For $\eta \leqslant \eta_2$, η decreases with increasing w. Furthermore, because of the relation given in Eq. (2.26), we have

$$\eta_{vis}(w) > \eta_{invis}(w) \qquad \text{for} \qquad w > 1 \ , \ V \leqslant 0 \ . \tag{2.31}$$

The above properties of the solution enable the solution curves to be sketched, as shown in Fig. 5. As the constant c decreases from zero, the minimum value decreases and the jump in w increases while the jump takes place farther upstream.

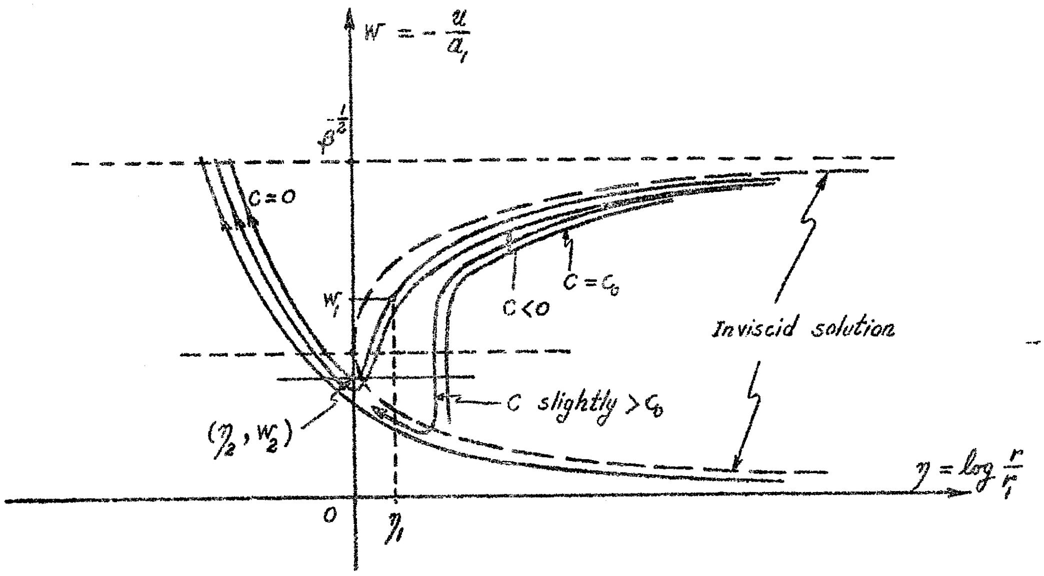

Fig. 5. Sketch of the Solution Curves in the Physical Space.

It follows from Eq. (2.27) that if we introduce

$$\eta = \alpha^{2/3}\xi \quad , \quad w = 1 + \alpha^{1/3}\tilde{w} \ , \tag{2.32a}$$

then Eq. (2.28) becomes

$$\frac{d^2\tilde{w}}{d\xi^2} = -2\tilde{w}\frac{d\tilde{w}}{d\xi} + (1-\beta) \tag{2.32b}$$

which is the equation governing the flow near $w = 1$, $\eta = 0$. Integrating this equation once, we obtain

$$\frac{d\tilde{w}}{d\xi} + \tilde{w}^2 = (1-\beta)\xi + \text{Const.} \tag{2.33}$$

This equation will be integrated and discussed in detail in our final calculation.

3. Calculation of the Solutions by Using PLK-Method*

In this section we shall calculate $w(\eta)$, $\theta(\eta)$ governed by the original system of Eqs. (1.15 - 1.16). Throughout this section μ will again be assumed constant so that $\bar{\mu} = 1$, but no restriction will be imposed on σ and k. Consequently Eqs. (1.15, 16) become

$$w^2 \frac{dw}{d\eta} + \frac{1}{\gamma}\left[w \frac{d\theta}{d\eta} - \theta \frac{dw}{d\eta} - \theta w\right] = -\frac{\gamma+1}{2\gamma}\alpha\left(\frac{d^2 w}{d\eta^2} - w\right) w^2 , \tag{3.1}$$

$$\theta + \frac{\gamma-1}{2}(1+b\alpha)w^2 + \frac{\gamma+1}{4\gamma}\alpha\left[\frac{1}{\sigma(1+k)}\frac{d\theta}{d\eta} + (\gamma-1)\frac{dw^2}{d\eta}\right] = \frac{\gamma+1}{2} \tag{3.2}$$

where

$$\alpha = \frac{4\gamma}{\gamma+1}(1+k)\bar{\alpha} \quad , \quad b = \frac{\gamma+1}{\gamma}\frac{k}{1+k} \quad . \tag{3.3}$$

Using the PLK-method, as described below, the generalization to the case $\mu = \mu(T)$ presents no particular difficulty (cf. § 7).

It was seen before that even for a simplified version of these equations, such as Eq. (2.6), conventional perturbation method merely leads to asymptotic solutions for small α near $w = 0$ or $w = \beta^{-\frac{1}{2}}$ (cf. Eqs. 2.10, 2.16) because coefficient of w^n does not diminish as $n \to \infty$. This asymptotic result fails to be a good approximation to the required solution as w deviates farther from $w = 0$ or $w = \beta^{-\frac{1}{2}}$ and becomes

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* Cf the footnote in the Introduction.

almost useless for calculations near $w = 1$. Now let us resort to the PLK-method, which is in essence to expand the solution in terms of power series in α with coefficients as undetermined functions of a parameter ξ,

$$\left.\begin{aligned} w(\xi) &= \xi + \alpha w^{(1)}(\xi) + \alpha^2 w^{(2)}(\xi) + \cdots \quad , \\ \eta(\xi) &= \eta^{(0)}(\xi) + \alpha \eta^{(1)}(\xi) + \alpha^2 \eta^{(2)}(\xi) + \cdots \quad , \\ \theta(\xi) &= \theta^{(0)}(\xi) + \alpha \theta^{(1)}(\xi) + \alpha^2 \theta^{(2)}(\xi) + \cdots \quad . \end{aligned}\right\} \tag{3.4}$$

The need of a parameter ξ to represent the solution and that w starts with the term ξ are clearly suggested by our previous discussions. Substituting these expansions into Eqs. (3.1) and (3.2), noting that

$$\frac{dw}{d\eta} = \frac{w'}{\eta'} \quad , \quad \frac{d\theta}{d\eta} = \frac{\theta'}{\eta'} \quad , \quad \frac{d^2 w}{d\eta^2} = \frac{w''}{(\eta')^2} - \frac{w'\eta''}{(\eta')^3} \tag{3.5}$$

where prime stands for $\frac{d}{d\xi}$, and then equating equal powers of α, we obtain the zeroth order equations as follows:

$$\left[\xi^2 + \frac{\xi^2}{\gamma}\frac{d}{d\xi}\left(\frac{\theta^{(0)}}{\xi}\right) - \frac{1}{\gamma}\xi\,\theta^{(0)}\eta^{(0)'}\right]\left(\eta^{(0)'}\right)^2 = 0 \tag{3.6a}$$

$$\left[\theta^{(0)} - \left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right)\right]\eta^{(0)'} = 0 \tag{3.6b}$$

If we choose $\eta^{(0)'}$ different from zero, then we have the zeroth order solution

$$\theta^{(0)}(\xi) = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2 \tag{3.7a}$$

and

$$\eta^{(0)}(\xi) = -\log\xi - \frac{1}{\gamma-1}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right| \tag{3.7b}$$

The integration constant in Eq. (3.7b) has been so chosen that when $\alpha \to 0$, $\xi \equiv w$ and $\eta^{(0)}(w)$ agrees with inviscid solution.

The coefficient of α in the expanded equations gives the following first order equations

$$\theta^{(1)} + (\gamma-1)\xi\, w^{(1)} = -\frac{\gamma-1}{2}\, b\xi^2 + \frac{\gamma+1}{2}\,\epsilon\, \frac{\xi^2(1-\beta\xi^2)}{1-\xi^2} \tag{3.8a}$$

and

$$\left(\eta^{(0)'}\right)^2 \Big\{ \left(2\xi + \frac{\theta^{(0)}}{\gamma}\right) w^{(1)} + \left(\xi^2 - \frac{\theta^{(0)}}{\gamma}\right) w^{(1)'} + \frac{1}{\gamma}\xi^2 \frac{d}{d\xi}\left(\frac{\theta^{(1)}}{\xi}\right)$$
$$-\frac{1}{\gamma}\left[\eta^{(0)'}\left(\xi\theta^{(1)} + w^{(1)}\theta^{(0)}\right) + \xi\theta^{(0)}\eta^{(1)'}\right]\Big\} = \frac{\gamma+1}{2\gamma}\xi^2\left[\eta^{(0)''} + \xi\left(\eta^{(0)'}\right)^3\right] \tag{3.8b}$$

where

$$b = \frac{\gamma+1}{\gamma}\,\frac{k}{1+k} \quad , \quad \mu' = 3k\mu + \mu, \quad \beta = \frac{\gamma-1}{\gamma+1} \quad , \quad \epsilon = \frac{\gamma-1}{\gamma}\left[1 - \frac{1}{2\sigma'(1+k)}\right] \tag{3.8c}$$

By substituting Eqs. (3.7) and (3.8a) into (3.8b), the terms in the curly bracket can be rearranged to take the following form

$$\frac{\gamma+1}{2\gamma}\,\xi(1-\beta\xi^2)\frac{d}{d\xi}\left[\eta^{(0)'}w^{(1)} - \eta^{(1)}\right] + \frac{\gamma+1}{2\gamma}\,\xi^2\eta^{(0)'}\left[b\beta\xi - \epsilon\,\frac{\xi(1-\beta\xi^2)}{1-\xi^2}\right]$$
$$-\frac{\gamma+1}{2\gamma}\left[b\beta - \epsilon\,\frac{1+(1-3\beta)\xi^2 + \beta\xi^4}{(1-\xi^2)^2}\right]\xi^2 \quad ,$$

and finally Eq. (3.8b) turns out to be

$$\frac{d}{d\xi}\left[\eta^{(0)'}w^{(1)} - \eta^{(1)}\right] = (1-\epsilon)\left[\frac{2\xi}{(1-\xi^2)^2} + \frac{1+\beta}{1-\beta}\,\frac{\xi}{\xi^2-1}\right] + \frac{(1-a)(1-\beta)}{\beta}\,\frac{\xi}{(\beta\xi^2-1)^2}$$
$$+\left[\frac{1-a}{\beta} - (a-\epsilon) - \frac{(1+\epsilon)2\beta}{1-\beta}\right]\frac{\xi}{(\beta\xi^2-1)} \quad , \tag{3.9}$$

where

$$a = \beta b = \frac{\gamma-1}{\gamma}\,\frac{k}{1+k}$$

Now in order to ascribe to $w^{(1)}$ a possibly lowest order singularity at $\xi = 1$ to improve the convergence of the series, we decompose Eq. (3.9) as follows

$$\frac{d}{d\xi}\left[\eta^{(0)'} w^{(1)}\right] = \frac{(1-a)(1-\beta)}{\beta}\,\frac{\xi}{(\beta\xi^2-1)^2} + \left[\frac{1-a}{\beta} - (a-\epsilon) - \frac{(1+\epsilon)2\beta}{1-\beta}\right]\frac{\xi}{(\beta\xi^2-1)}\ , \tag{3.10a}$$

$$\frac{d\eta^{(1)}}{d\xi} = -(1-\epsilon)\left[\frac{2\xi}{(1-\xi^2)^2} + \frac{\gamma\xi}{(\xi^2-1)}\right] . \tag{3.10b}$$

The solution of Eqs. (3.10a, b) is thus

$$w^{(1)}(\xi) = -A\,\frac{\xi}{1-\xi^2} - B\,\frac{\xi(1-\beta\xi^2)}{1-\xi^2}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right| \quad , \tag{3.11a}$$

$$\eta^{(1)}(\xi) = -(1-\epsilon)\left[\frac{1}{1-\xi^2} + \frac{\gamma}{2}\log\left|1-\xi^2\right|\right] \quad , \tag{3.11b}$$

where

$$A = \frac{(1-a)(1-\beta)}{2\beta^2} \quad , \qquad B = \frac{1}{2\beta}\left[\frac{1-a}{\beta} - (a-\epsilon) - \frac{(1+\epsilon)2\beta}{1-\beta}\right] \tag{3.11c}$$

In the above first order solution, $w^{(1)}$ and $\eta^{(1)}$ have singularities at $\xi = 1$ of the same order. After having determined $w^{(1)}$, $\theta^{(1)}(\xi)$ is then given by Eq. (3.8a).

Proceeding in a similar manner to obtain the second-order equations by equating terms with α^2, we find that the resulting equations possess the solution of quite lengthy expression, in which $\eta^{(2)}(\xi)$ starts with the term $2(1-\beta)(1-\epsilon)\frac{1}{(1-\xi^2)^4}$ followed by terms of $O\left(\frac{1}{(1-\xi^2)^3}\right)$ while $w^{(2)}(\xi)$ still can be made to be of $O\left(\frac{1}{(1-\xi^2)}\right)$. Therefore the final solution can be expressed parametrically as follows:

$$w(\xi) = \xi - \alpha\left[A\,\frac{\xi}{1-\xi^2} + B\,\frac{\xi(1-\beta\xi^2)}{1-\xi^2}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right] + O\left(\frac{\alpha^2}{1-\xi^2}\right) \tag{3.12a}$$

$$\begin{aligned}\eta(\xi) = &-\left[\log\xi + \frac{1}{\gamma-1}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right] - \alpha(1-\epsilon)\left[\frac{1}{1-\xi^2} + \frac{\gamma}{2}\log|1-\xi^2|\right] \\ &+ \alpha^2\,\frac{2(1-\beta)(1-\epsilon)}{(1-\xi^2)^4} + O\left(\frac{\alpha^2}{(1-\xi^2)^3}, \frac{\alpha^3}{(1-\xi^2)^7}\right) \\ &+ C\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi\right|^{\frac{(1-\beta)^2}{2\alpha\beta}}\end{aligned} \tag{3.12b}$$

where A, B, ϵ are constants given in Eqs. (3.11c), (3.8c) and

$$\left.\begin{aligned}&C = 0 \quad \text{for the solution starting from } \xi = 0\,, \\ &C \quad \text{is arbitrary for the solution starting from } \xi = \beta^{-\frac{1}{2}}\,.\end{aligned}\right\} \tag{3.12c}$$

The value of θ is given by

$$\begin{aligned}\theta(\xi) = &\left(\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right) + \alpha(\gamma-1)\left[A\,\frac{\xi^2}{1-\xi^2} + B\,\frac{\xi^2(1-\beta\xi^2)}{1-\xi^2}\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right. \\ &\left. - \frac{b}{2}\xi^2 + \frac{1}{2\beta}\,\epsilon\,\frac{\xi^2(1-\beta\xi^2)}{1-\xi^2}\right] + O\left(\frac{\alpha^2}{(1-\xi^2)^2}\right).\end{aligned} \tag{3.12d}$$

Several interesting features of the above solution may be mentioned here. (i) $\epsilon = 0$ for $\sigma = \frac{1}{2(1+k)}$ ($= \frac{3}{4}$ if $k = -\frac{1}{3}$, cf. Eq. 3.8c). With the value of σ and k lying in the experimental range, ϵ is still a small number. Consequently, the variation in σ and k only contribute a small correction to the coefficients of $O(1)$ in the solution. This fact confirms our previous statement in § 2.1. (ii) By substituting Eq. (3.12) into Eq. (2.2), it can be found that the arbitrary constant $\tilde{A}$ is of $O(\alpha\epsilon)$. (iii) The most important property of the above solution is that they do not provide an approximate solution with uniform accuracy over the interval of ξ such that $0 \leq w \leq \beta^{-\frac{1}{2}}$. As ξ approaches unity, the higher order terms, especially in $\eta(\xi)$, become more important relative to the zeroth order term. More precisely, the solution is good only for $0 \leq \xi \leq 1 - K\alpha^{1/3}$ and $1 + K\alpha^{1/3} \leq \xi \leq \beta^{-\frac{1}{2}}$, K being a constant of $O(1)$. At $\xi = 1 \pm K\alpha^{1/3}$, all terms in the expression for $\eta(\xi)$ become of the same order, $O(\alpha^{2/3})$; but the convergence can be made sufficiently rapid by an appropriate choice of the value K.

In the subsequent calculation of the solution through the transonic region, we shall only consider a particular solution with $C = 0$, $\epsilon = 0$ in Eqs. (3.12). Furthermore, it has been found convenient to take $K = 2(\gamma+1)^{1/3}$. With this value of K, we obtain, from Eqs. (3.12a, b), the following result:

(i) supersonic branch,

$$\text{at} \quad w = 1 + 2\left(\frac{\alpha}{\gamma+1}\right)^{1/3}, \quad \eta = 2.281(\gamma+1)^{1/3}\alpha^{2/3} \quad \text{and} \quad \frac{dw}{d\eta} = 0.615\frac{\alpha^{-\frac{1}{3}}}{(\gamma+1)^{2/3}}; \tag{3.13a}$$

(ii) subsonic branch

$$\text{at} \quad w = 1 - 2\left(\frac{\alpha}{\gamma+1}\right)^{1/3}, \quad \eta = 1.766(\gamma+1)^{1/3}\alpha^{2/3} \quad \text{and} \quad \frac{dw}{d\eta} = -0.478\frac{\alpha^{-\frac{1}{3}}}{(\gamma+1)^{2/3}}. \tag{3.13b}$$

These values will serve for the boundary conditions imposed on the transonic solution to be obtained below. That the PLK method is powerful in solving this problem can still further be stressed by the following argument. As the first order term in the expression for $w(\xi)$ (cf. 3.12a) is quite unimportant in the aforementioned regions for ξ, one perhaps would try, instead of Eq. (3.4), a simpler expansion

$$\eta(w) = \eta^{(0)}(w) + \alpha\,\eta^{(1)}(w) + \cdots \tag{3.14}$$

and a similar expansion for θ in terms of w. It can be shown that the above expansion will yield a solution in which $\eta^{(0)}$ is identical to inviscid solution, but $\eta^{(1)}(w)$ has, in addition to a simple pole at $w=1$, a pole and a logarithmic singularity at $w=\beta^{-\frac{1}{2}}$. Consequently the assumed expansion (3.14) becomes invalid for r large on the supersonic branch, and thus leads to an erroneous result.

4. The Solution in the Transonic Region

To obtain an approximate solution in the transonic region, as discussed in § 2.2(d), (e) (cf. Eqs. 2.27, 28 and 32) and also as guided by the boundary conditions Eq. (3.13), first we distort the independent variable by the transformation

$$\eta = \alpha^{2/3}\xi \quad , \tag{4.1}$$

and then expand w, θ into the form

$$w(\xi) = 1 + \alpha^{1/3} w^{(1)}(\xi) + \alpha^{2/3} w^{(2)}(\xi) + \alpha w^{(3)}(\xi) + \cdots \quad , \tag{4.2}$$

$$\theta(\xi) = 1 + \alpha^{1/3}\theta^{(1)}(\xi) + \alpha^{2/3}\theta^{(2)}(\xi) + \alpha\theta^{(3)}(\xi) + \cdots \quad . \tag{4.3}$$

With $\epsilon = 0$ $(\sim 2\sigma(1+k) = 1)$, Eqs. (3.1) and (3.2) reduce to

$$\left[1-(1-a\alpha)w^2\right]\frac{dw}{d\eta} + w\left\{1-\left[\beta+(a-1)\alpha\right]w^2\right\} = \alpha w^2 \frac{d^2w}{d\eta^2} \tag{4.4}$$

$$\theta = \frac{\gamma+1}{2} - \frac{\gamma-1}{2}(1+b\alpha)w^2 + O(\alpha^2) \tag{4.5}$$

Substituting Eqs. (4.1) - (4.3) into Eqs. (4.4) and (4.5), we obtain the first order equation:

$$\frac{d^2w^{(1)}}{d\xi^2} + 2w^{(1)}\frac{dw^{(1)}}{d\xi} = (1-\beta) \tag{4.6}$$

and the second order equation:

$$\frac{d^2 w^{(2)}}{d\xi^2} + 2\frac{d}{d\xi}\left(w^{(1)} w^{(2)}\right) = \frac{d}{d\xi}\left(w^{(1)^3}\right) - (1+\beta)\, w^{(1)} . \tag{4.7}$$

The correction due to the terms with constants a and b enters only in $w^{(3)}$, $\theta^{(3)}$ and higher order terms.

Now Eq. (4.6) can be integrated once to yield:

$$\frac{dw^{(1)}}{d\xi} + w^{(1)^2} = \frac{2}{\gamma+1}\xi + D_1 \tag{4.8}$$

where the constant D_1 can be determined by using the boundary condition (3.13); the result is,

$$\left.\begin{aligned} D_1 &= 0.053\,(\gamma+1)^{-2/3} && \text{for the subsonic branch,} \\ D_1 &= -0.01\,(\gamma+1)^{-2/3} && \text{for the supersonic branch.} \end{aligned}\right\} \tag{4.9}$$

It is convenient to rewrite Eq. (4.8) in the following form

$$\frac{dy}{dx} + y^2 = x + x_1 \tag{4.10}$$

where

$$y = \left(\frac{\gamma+1}{2}\right)^{1/3} w^{(1)} \quad , \quad x = \left(\frac{2}{\gamma+1}\right)^{1/3}\xi \quad , \quad x_1 = \left(\frac{\gamma+1}{2}\right)^{2/3} D_1 \quad . \tag{4.10a}$$

Now Eq. (4.10) is of the Riccati type, which, by the transformation

$$y(x) = \frac{1}{v}\frac{dv}{dx} \quad , \tag{4.11}$$

can be reduced to a second order linear equation:

$$\frac{d^2 v}{dx^2} - (x + x_1)\, v = 0 \tag{4.12}$$

The solution of this equation for $(x+x_1) > 0$ is

$$v = M z^{1/3}\left[I_{-\frac{1}{3}}(z) + N I_{\frac{1}{3}}(z) \right] \quad , \quad z = \frac{2}{3}(x+x_1)^{3/2} \quad , \tag{4.13}$$

where $I(z)$ is the modified Bessel function of the first kind, and M , N

are the integration constants. By using Eq. (4.11), the solution of Eq. (4.10) is then

$$y(z) = \left(\frac{3z}{2}\right)^{1/3} \frac{I_{2/3}(z) + N I_{-2/3}(z)}{I_{-1/3}(z) + N I_{1/3}(z)} \quad , \tag{4.14}$$

where

$$z = \frac{2}{3}\,(x+x_1)^{3/2}, \qquad \text{and} \qquad x+x_1 > 0 \quad . \tag{4.14a}$$

The constant N can be determined by using the condition (3.13).

The continuation of Eq. (4.14) into the region $x+x_1<0$ is provided by

$$\left.\begin{aligned} z^{1/3} I_{1/3}(z) &= -\zeta^{1/3} J_{\frac{1}{3}}(\zeta) \quad , & z^{1/3} I_{-\frac{1}{3}}(z) &= \zeta^{1/3} J_{-\frac{1}{3}}(\zeta) \\ z^{2/3} I_{2/3}(z) &= \zeta^{2/3} J_{\frac{2}{3}}(\zeta) \quad , & z^{2/3} I_{-2/3}(z) &= \zeta^{2/3} J_{-2/3}(\zeta) \end{aligned}\right\} \tag{4.15}$$

where

$$\zeta = z\, e^{-3\pi i/2} = \frac{2}{3}\left[-(x+x_1)\right]^{3/2}$$

Consequently Eq. (4.14) becomes

$$y(\zeta) = \left(\frac{3\zeta}{2}\right)^{1/3} \frac{J_{2/3}(\zeta) + N J_{-2/3}(\zeta)}{J_{-1/3}(\zeta) - N J_{1/3}(\zeta)} \qquad \text{for} \qquad x+x_1 < 0 \,. \tag{4.16}$$

To discuss the above solution, we first note that the inviscid solution in this transonic region is

$$y^2 = x \tag{4.17}$$

which has two branches for $x>0$ and gives no solution for $x<0$. Now before we determine the value of N for the corresponding viscous solutions, we may also note that the general solution, given by (4.14) and (4.16), is a semi-trancendental function of D_1 and the second integration constant N. It can be shown, from the properties of $I_\nu(z)$

at large z , that in Eq. (4.14)

$$y \to +\sqrt{x} \quad \text{as} \quad x \to \infty \qquad N > -1 \quad ,$$

$$y \to -\sqrt{x} \quad \text{as} \quad x \to \infty \qquad N = -1 \quad ,$$

and y has a simple pole at a certain finite x for $N<-1$ (which is of no physical significance). This result shows that the viscous solutions tend to their respective inviscid values for x large in a manner which implies again that $(W=\beta^{-\frac{1}{2}}, \eta=\infty)$ is a nodal point (admitting more than one value of N) while $(W=0, \eta=\infty)$ is a saddle point (admitting only one value of N). However, for $x+x_1<0$, Eq. (4.16) shows obviously that $y(\zeta)$ has infinite number of isolated simple poles at ζ_n where the denominator vanishes. Since the properties of the solution curves in the (W,V) phase space exhibit no such singularities, the solution (4.16), therefore, presents a good approximation to the real flow only for ζ lying in the interval $0\leqslant\zeta\leqslant\zeta_1-\delta<\zeta_1$ where ζ_1 is the first pole and δ is a positive number, appropriately chosen such that $y(\zeta_1-\delta)$ is not yet too large to void our approximation (4.2).

Having obtained the first order solution $W^{(1)}(\xi)=\left(\frac{2}{\gamma+1}\right)^{1/3} y$, the second order equation (4.7) can be then integrated to yield

$$W^{(2)}(\xi) = e^{-2\varphi(\xi)}\int \psi(\xi)\, e^{2\varphi(\xi)}\, d\xi \tag{4.18}$$

where

$$\varphi(\xi)=\int W^{(1)}(\xi)\, d\xi \qquad \text{and} \qquad \psi(\xi) = \left[W^{(1)}(\xi)\right]^3 - (1+\beta)\,\varphi(\xi) .$$

It is obvious that $W^{(2)}(\xi)$ is bounded wherever $W^{(1)}(\xi)$ is bounded. Consequently, the approximation is good even if we only take the first two terms in (4.2) and (4.3).

In order to obtain some numerical results, we first determine the value N in Eqs. (4.14), (4.16) by using conditions (3.13) to obtain

$$\left.\begin{array}{ll} N=-0.585 & \text{for the supersonic branch,} \\ N=-1 & \text{for the subsonic branch.} \end{array}\right\} \quad (4.19)$$

With these values of D, and N (cf. 4.9 and 4.19), the solutions are plotted in Fig. 6 (by using tables, Ref. 8) from which several interesting results can be deduced as follows:

(i) For the supersonic branch, the transonic solution starts from point A (cf. Fig. 6) with the coordinates

$$W_1 \doteq 1+1.6\left(\frac{2}{\gamma+1}\right)^{1/3}\alpha^{1/3} \quad , \quad \eta_1 \doteq 2.9\left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{2/3} \quad (4.20)$$

After the solution curve passes through a point of inflection G and then crosses the line $y=0$ $(W=1)$ at point B (cf. also Fig. 4) where $\eta_B \doteq 1.02\left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{2/3}$ $\left(\sim V_B \doteq -1.05\left(\frac{2}{\gamma+1}\right)^{2/3}\alpha^{-\frac{1}{3}}\right.$ and $\left.\left(\frac{dV}{dW}\right)_B \doteq -0.95\left(\frac{2}{\gamma+1}\right)^{2/3}\alpha^{-2/3}\right)$, it reaches a minimum when it intercepts the curve $y^2=x$ at C where

$$W_2 \doteq 1-0.45\left(\frac{2}{\gamma+1}\right)^{1/3}\alpha^{1/3} \quad , \quad \eta_2 \doteq 0.2\left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{2/3} \quad . \quad (4.21)$$

It then increases from $W=W_2$ to $W=1$ at point D where $\eta_D \doteq -0.88\left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{2/3}$ $\left(\sim V_D \doteq 0.908\left(\frac{2}{\gamma+1}\right)^{2/3}\alpha^{-\frac{1}{3}}, \ \frac{dV}{dW}=1.1\left(\frac{2}{\gamma+1}\right)^{2/3}\alpha^{-2/3}\right)$.

That is, there is an expansion wave following the cylindrical shock.

(ii) The thickness of the cylindrical shock, as defined in § 2.2(d), is

$$\Delta\eta=\eta_1-\eta_2=2.7\left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{2/3} \quad (\text{in } r\text{-space}, \ \Delta r \doteq r_1\Delta\eta) \quad (4.22)$$

across which the velocity has a "jump"

$$\Delta W=W_1-W_2=2.05\left(\frac{2}{\gamma+1}\right)^{1/3}\alpha^{1/3} \quad (4.23)$$

and

$$W_1 W_2 = 1 + 1.15\left(\frac{2}{\gamma+1}\right)^{1/3} \alpha^{1/3} . \qquad (4.24)$$

Combining Eqs. (4.22) and (4.23), we obtain

$$\Delta\eta \doteq 5.5 \frac{\alpha}{\Delta w} . \qquad (4.25)$$

Comparing these results with those of a plane normal shock (e.g. Ref. 9), we note first that the plane shock strength $(\sim \Delta w)$ is quite arbitrary while for a cylindrical shock, $\Delta w \sim O(\alpha^{1/3})$. The expression for shock thickness (Eq. 4.22) shows that $\Delta\eta \sim O(\alpha^{2/3})$, although, combining Δw, the expression (4.25) agrees with that of a plane shock (cf. Ref. 9) within the order of magnitude. The result (4.25) differs, however, from Levey's result for the diffuse shock in a source flow (Ref. 4 Eq. 4.9), in which he explains the discrepancy as due to some degree of choice of the definition of the shock thickness. Our result also indicates that the maximum velocity gradient inside a cylindrical shock is of order $\alpha^{-1/3}$, (in contrast to Levey's result: $O(\alpha^{-1})$), while for a plane shock, the maximum gradient is (cf. Ref. 9) of order $(\Delta w)^2 \alpha^{-1}$ which reduces to $O(\alpha^{-1/3})$ if $\Delta w \sim O(\alpha^{1/3})$. The expression (4.24) differs from the Prandtl relation of a plane shock by a term of $O(\alpha^{1/3})$ which here agrees with Levey's result (Ref. 4, Eq. 4.13). The differences between the present results of cylindrical flow and those of one-dimensional plane shock can perhaps be realized by visualizing that the viscous forces exerting on the surfaces $r d\theta$ and dr of a fluid element is indeed of quite different nature from those exerting in plane shock flow, since in the former case, the normal stress acting on the surface dr will have a component in the radial direction.

(iii) On the subsonic branch, w is a monotonic decreasing function of η. $w=1$ at point E (cf. Figs. 6 and 4) where $\eta_E = -1.02\left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{2/3}$ and the velocity gradient $\left(\frac{dw}{d\eta}\right)_E = -V_E = -1.014\left(\frac{2}{\gamma+1}\right)^{2/3}\alpha^{-1/3}$ (cf. also Eq. 2.21a), which shows that the solution curve of the subsonic branch passes $w=1$ (cf. Fig. 4) slightly above the curve C_2.

(iv) The thermodynamic variables in this flow region can be deduced from Eq. (4.5), (1.3) and (1.4). That is, in the expansion (4.3) and

$$\left.\begin{aligned}\bar{p}(\xi) &= 1+\alpha^{1/3}p^{(1)}(\xi)+\alpha^{2/3}p^{(2)}(\xi)+\cdots\\ \bar{\rho}(\xi) &= 1+\alpha^{1/3}\rho^{(1)}(\xi)+\alpha^{2/3}\rho^{(2)}(\xi)+\cdots\end{aligned}\right\}\tag{4.26}$$

we have

$$\theta^{(1)}(\xi) = -(\gamma-1)\left(\frac{2}{\gamma+1}\right)^{1/3} y(\xi)\tag{4.27}$$

and

$$p^{(1)} = \gamma\rho^{(1)} = \frac{\gamma}{\gamma-1}\,\theta^{(1)}\tag{4.28}$$

where $y(\xi)$ is given in Eqs. (4.14) and (4.16). The values of $\rho^{(1)}$ and $\theta^{(1)}$ vs. η is plotted in Fig. 7. The supersonic branch starts with compression and is then followed by an expansion wave, while the subsonic branch expands continuously. Eq. (4.28) states simply that $\bar{p}$, $\bar{\rho}$ and θ satisfy the isentropic relation up to $O(\alpha^{1/3})$. This implies that the entropy variation, if any, across this region must be of order at least $\alpha^{+\frac{2}{3}}$.

5. The Solution in the Inner Supersonic Region

The above transonic solution shows that the supersonic branch flow approaches asymptotically to the subsonic branch and for $\eta < \eta_E$ both branches have supersonic local speed. We shall proceed to find the solution for $\eta < \eta_E$. Let us consider first the continuation of the subsonic branch. For the sake of convenience, we shall take the point in Fig. 6 as the boundary condition such that

$$\left.\begin{aligned} & w_E = 1 , \quad \eta_E = -1.02\left(\frac{\gamma+1}{2}\right)^{1/3} \alpha^{2/3} , \quad V_E = -\left(\frac{dw}{d\eta}\right)_E = 1.014\left(\frac{2}{\gamma+1}\right)^{2/3} \alpha^{-\frac{1}{3}} \\ & \text{and} \qquad \left(\frac{dV}{dw}\right)_E = 0.985\left(\frac{2}{\gamma+1}\right)^{2/3} \alpha^{-2/3} \end{aligned}\right\} \tag{5.1}$$

To obtain the solution for $\eta < \eta_E$, one can try the following expansion

$$\left.\begin{aligned} w &= \frac{1}{\sqrt{\beta}} + x \\ \eta(x) &= \alpha\,\eta^{(0)}(x) + \alpha^2 \eta^{(1)}(x) + \alpha^3 \eta^{(2)}(x) + \cdots \\ \theta(x) &= \theta^{(0)}(x) + \alpha\,\theta^{(1)}(x) + \cdots \end{aligned}\right\} \tag{5.2}$$

which is to be substituted into Eqs. (3.1) and (3.2). However, by noting the boundary condition (5.1), a more convenient method to approximate the solution can be carried out by letting

$$V = -\frac{dt}{d\eta} = a_0 \alpha^{-\frac{1}{3}} + a_1 \alpha^{-\frac{2}{3}} t + (a_2/2)\alpha^{-1} t^2 \qquad \text{for} \qquad t = w - 1 > 0 \tag{5.3}$$

where

$$a_0 = 1.014\left(\frac{2}{\gamma+1}\right)^{2/3} , \quad a_1 = 0.985\left(\frac{2}{\gamma+1}\right)^{2/3} , \quad a_2 = \sqrt{\beta}\,(1+\sqrt{\beta}) \tag{5.3a}$$

so that the conditions (5.1) are satisfied and, in addition $\left(\frac{dV}{dw}\right)$ will take the value $\frac{1-\beta}{\alpha}$ at $w = \beta^{-\frac{1}{2}}$ (cf. § 2.2a). Integrating (5.3) and using Eq. (5.1), we obtain

$$\eta = \eta_E - \frac{2\alpha^{2/3}}{\sqrt{2a_0a_2 - a_1^2}} \tan^{-1}\left(\frac{\sqrt{\frac{2a_0a_2}{a_1^3} - 1}\; t}{t + \frac{2a_0}{a_1}\alpha^{1/3}} \right) \quad , \text{ for } \quad t > 0 \qquad (5.4)$$

This solution is in good agreement with the Eq. (4.16) for $\eta_E > \eta > \eta_F$. It can be seen at once that as $w \to \beta^{-\frac{1}{2}}$ (the maximum velocity at which $\theta = 0$), or $t \to \beta^{-\frac{1}{2}} - 1$, η approaches its smallest value η_m , say, where

$$\eta_m - \eta_E = -1.95 \left(\frac{\gamma+1}{2} \right)^{1/3} \alpha^{2/3} \qquad (5.5)$$

in which the assigned values of a_0 , a_1 , a_2 have been used. Thus we see that the flow supposedly terminates itself at a distance of $O(\alpha^{2/3})$ to the inner side of $\eta = 0$, beyond which there is no solution to our present system of equations. To search further for the possibilities whether one still could obtain the solution of physical reality for $\eta < \eta_m$, one would face some rather dubious situations. For instance, near $\eta = \eta_m$, the density, temperature and pressure all become so low that the validity of the equation of state for perfect gas (1.4) is questionable. Besides, the fact that the viscous stresses reach the magnitude of the fluid pressure near $\eta = \eta_m$ sets a likely limit as to the applicability of the Navier - Stokes equation (1.1) and also raises a question as to whether Burnett's higher viscous terms (Ref. 10, p. 271) should be employed to overcome the present difficulty. Of course, it would seem plausible to continue our solution further inward by assigning appropriate values to the arbitrary constant C in (2.16). Nevertheless, it is still impossible to bring the flow to $\eta = -\infty$ $(r = 0)$ on account of the singularity that $\rho u \sim r^{-1}$ near $r = 0$ (cf. 1.3). To clarify these rather vague points is beyond the scope of this paper, although such clarification is certainly desirable.

6. The Entropy Variation

We define S to be the specific entropy,

$$T\,dS = c_p\,dT - \frac{1}{\rho}\,dp \qquad (6.1)$$

then the energy equation (1.2) can be written as

$$\rho T u \frac{dS}{dr} = \mathrm{div}(\lambda\,\mathrm{grad}\,T) + \Phi \qquad (6.2)$$

where Φ is the viscous dissipation function, which in this case is,

$$\Phi = \frac{2}{3}(\mu' + 2\mu)\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{u}{r}\right)^2\right] + \frac{4}{3}(\mu' - \mu)\frac{u}{r}\frac{\partial u}{\partial r} \,. \qquad (6.3)$$

The above definition of S of a fluid element is clearly for an open system since the heat exchange by conduction, and hence a net flow of entropy, occurs with the neighboring elements. Thus Eq. (6.2) merely expresses the energy balance, in terms of S, of the fluid element — a system not isolated, in the thermodynamic sense, from its surroundings. The analysis of formulating the second law of thermodynamics for the fluid flow case by making the system closed has been investigated in some detail by Tolman and Fine (Ref. 11) and discussed later by Curtiss and Hirschfelder (Ref. 12) from the point of view of statistical mechanics. Their idea is in essence to state that the change ΔS in the entropy of a system should be constituted of not only the entropy carried into the system due to conduction of heat energy, equal to $\mathrm{div}\left(\frac{\lambda}{T}\,\mathrm{grad}\,T\right)$ per unit volume, but also the net increase in entropy produced by irreversible processes taking place inside the system. Following Tolman's notation, we may thus write

$$\rho\left(\frac{DS}{Dt}\right)_{irr} = \rho\frac{DS}{Dt} - \mathrm{div}\left(\frac{\lambda}{T}\,\mathrm{grad}\,T\right) \,. \qquad (6.4)$$

In the present case, Eq. (6.4), after combining with (6.2), becomes

$$\rho T u \left(\frac{dS}{dr}\right)_{irr} = \frac{\lambda}{T}(\text{grad } T)^2 + \Phi \tag{6.5}$$

As we are only interested in the qualitative features of our later results, we simplify these equations by using the assumptions

$$\bar{\mu} = \mu/\mu_1 = 1 \quad , \quad \mu' = 0 \quad , \quad C_p, C_v = \text{constant} \quad , \sigma = \frac{3}{4} \quad . \tag{6.6}$$

Then the nondimensional form of Eqs. (6.2) and (6.5) are respectively

$$-\frac{\theta}{\gamma-1}\frac{ds}{d\eta} = \bar{\alpha}\frac{4\gamma}{3}\left\{\frac{1}{(\gamma-1)}\frac{d^2\theta}{d\eta^2} + \left(\frac{dw}{d\eta}\right)^2 - w\frac{dw}{d\eta} + w^2\right\} \tag{6.7}$$

where

$$s = S/C_v \tag{6.7a}$$

and

$$-\frac{\theta}{\gamma-1}\left(\frac{ds}{d\eta}\right)_{irr} = \frac{4\gamma}{3}\bar{\alpha}\left\{\frac{1}{(\gamma-1)}\frac{1}{\theta}\left(\frac{d\theta}{d\eta}\right)^2 + \left(\frac{dw}{d\eta}\right)^2 - w\frac{dw}{d\eta} + w^2\right\} \tag{6.8}$$

Though the sign of the terms on the right hand side of (6.7) is in general indefinite, the value of the right hand side terms of (6.8) is, however, positive definite. Therefore $(s)_{irr}$ increases monotonically along the fluid flow, as predicted by the second law for a closed system. Subtracting Eq. (6.8) from (6.7), we obtain

$$\left(\frac{ds}{d\eta}\right)_{irr} = \frac{ds}{d\eta} + \frac{4\gamma}{3}\bar{\alpha}\frac{d^2}{d\eta^2}\log\theta$$

which can be integrated to yield

$$s_{irr} = s + \frac{4\gamma}{3}\bar{\alpha}\frac{d\log\theta}{d\eta} \tag{6.9}$$

where the constant of integration is so chosen that both s and s_{irr} tend to s_0 as $\eta \rightarrow \infty$, s_0 being arbitrary.

In order to see that s of the shock type flow reaches a maximum near $w=1$, we substitute Eq. (2.3) into (6.7) and obtain

$$\frac{\theta}{\gamma-1}\frac{ds}{d\eta} = \frac{4\gamma}{3}\bar{\alpha}\, w\left(\frac{d^2w}{d\eta^2} + \frac{dw}{d\eta} - w\right) \quad . \tag{6.10}$$

This equation shows that for η outside the transonic region, the variation in s is at most of $O(\alpha)$. Within the transonic region, $\frac{d^2w}{d\eta^2}$, being of $O(\alpha^{-1})$, overwhelms the rest of the terms in the bracket and hence (6.10) reduces to

$$\frac{\theta}{\gamma-1}\frac{ds}{d\eta} = \frac{4\gamma}{3}\bar{\alpha}\, w\,\frac{d^2w}{d\eta^2}\left(1+O(\alpha^{2/3})\right) \quad . \tag{6.11}$$

It then follows that s assumes its maximum value at the point where the curvature of $w=w(\eta)$ curve vanishes ($\frac{d^2w}{d\eta^2}=0$ at point G in Fig. 6 and at this point $\frac{d^2s}{d\eta^2}$ is less than zero). However, from (6.9), the quantity $s+\frac{4\gamma}{3}\bar{\alpha}\frac{d}{d\eta}\log\theta$ does not have an extremum in the entire flow region. The above result is very much the same as that of a plane shock (e.g. Ref. 9), the solution of which shows that the velocity has a point of inflection at $w=1$ where the entropy is also maximum.

Integrating Eq. (6.10) with the aid of Eqs. (3.1) and (3.2) under condition (6.6), we obtain

$$s-s_0 = \log\left[\theta\,(w\bar{r})^{(\gamma-1)}\right] = \log\theta + (\gamma-1)(\log w+\eta) \tag{6.12}$$

where
$$s_0 = \log\left(p_0/\rho_0^{\gamma}\right)$$

so that $s \to s_0$ as $\bar{r}\to\infty$. This equation is actually the definition of s usually given for a perfect gas. Substitution of the solution (3.12) into (6.12) shows that

$$\Delta S \sim O(\alpha) \qquad \text{for} \qquad \eta > O(1) \tag{6.13}$$

Within and around the transonic region, we substitute the solution (4.1)-

(4.3) into (6.12) and simplify the expansion, then we obtain

$$s - s_o = \alpha^{2/3}(\gamma-1)\left(\frac{\gamma+1}{2}\right)^{1/3}\left[x - y^2\right] + O(\alpha) \tag{6.14}$$

where

$$x = \left(\frac{2}{\gamma+1}\right)^{1/3}\alpha^{-2/3}\eta \quad , \quad y = \left(\frac{\gamma+1}{2}\right)^{1/3}\alpha^{-\frac{1}{3}}(W-1) \tag{6.14a}$$

and the value $y = y(x)$ is given by Eqs. (4.14), (4.16) and also plotted in Fig. 6. Equation (6.14) is consistent with the fact that $s = s_o$ = const. along the inviscid solution $y^2 = x$. The variation in s along supersonic and subsonic branches of our solution follows directly from the data shown in Fig. 6. The result is plotted in Fig. 8. As η decreases along the supersonic branch, the entropy s first increases till it reaches the maximum $s_o + 1.21\,\alpha^{2/3}\left(\frac{\gamma+1}{2}\right)^{1/3}(\gamma-1)$ at point G, then decreases and later assumes once again the value s_o (the value of s at $\eta = \infty$) at point C where $W^{(1)}$ is minimum there. After that s decreases rapidly with further decrease in η and eventually tend to $-\infty$ as the flow solution terminates. On the subsonic branch, s decreases monotically with decreasing η. However, by substituting Eqs. (6.14) and (4.3) into (6.9), it can easily be shown that (s_{irr}) increases monotically with decreasing η and the variation in (s_{irr}) is of order $O(\alpha)$. Consequently, the result that $s \to -\infty$ as $\eta \to \eta_{min}$ can be explained by visualizing from Eq. (6.9) that $\frac{d}{d\eta}(\log\theta)$ decreases beyond all bounds as $\eta \to \eta_{min}$. Physically, this probably implies that the flow is rather far from its equilibrium condition due to the large velocity gradient, inducing a rapid decrease in temperature which even the important heat conduction in this region can not compensate.

7. The Case when μ Proportional to Temperature

In our previous investigation, μ , μ' and λ were assumed to be constant. In this section the effects on the solution due to the variation of μ , μ' and λ , all assumed to be proportional to T , will be calculated and compared with the previous results. That is,

$$\mu = const.\, T \quad , \quad \text{then} \quad \bar{\mu} = \theta \quad . \tag{7.1}$$

The dependence of μ on T other than the above relation can be worked out in a similar manner. In the present case $\bar{\alpha}$ depends on T_1 only, but not on the local temperature T . Besides, μ , μ' and λ all have the same dependence on T . Hence, assuming c_p , c_v and γ still to be constant, we may take

$$\bar{\alpha} \;,\; \alpha \;,\; k = \mu'/\mu \;,\quad \sigma = c_p \mu/\lambda \qquad \text{all constant.} \tag{7.2}$$

Then the fundamental system, Eqs. (1.15), (1.16), becomes

$$\frac{dw}{d\eta} + \frac{1}{\gamma}\left[\frac{d}{d\eta}\left(\frac{\theta}{w}\right) - \frac{\theta}{w}\right] = -\frac{\gamma+1}{2\gamma}\alpha\left\{\theta\left(\frac{d^2 w}{d\eta^2} - w\right) + \left[\frac{dw}{d\eta} + \frac{k}{1+k}w\right]\frac{d\theta}{d\eta}\right\} \tag{7.3}$$

and

$$\frac{w^2}{2} + \frac{\theta}{\gamma-1} + \frac{\gamma+1}{4\gamma}\alpha\theta\left[\frac{dw^2}{d\eta} + \frac{1}{(\gamma-1)\sigma(1+k)}\frac{d\theta}{d\eta} + \frac{2k}{1+k}w^2\right] = \frac{\gamma+1}{2(\gamma-1)} \quad . \tag{7.4}$$

In the outer region, we substitute the same expansion Eq. (3.4) into Eqs. (7.3) and (7.4), then carry out the calculation in a way similar to that described in § 3. The result is found to be

$$w(\xi) = \xi + \alpha\left(\frac{\gamma+1}{2}\right)\frac{\xi(1-\beta\xi^2)}{1-\xi^2}\left[A'\xi^2 + B'\log\left|\frac{\gamma+1}{2} - \frac{\gamma-1}{2}\xi^2\right|\right] + O\left(\frac{\alpha^2}{1-\xi^2}\right) \tag{7.5}$$

where

$$A' = \frac{1-\sigma}{2\beta} - \frac{3\sigma-\epsilon}{2} - \frac{3}{2}\beta(1-\epsilon) + \frac{k}{1+k} \quad , \quad B' = \frac{(1-\sigma)(1-\beta)}{2\beta^2} - \frac{1}{\beta}\left(\sigma - \frac{k}{1+k}\right) \quad ; \tag{7.5a}$$

and $\eta(\xi)$ has the same expression as Eq. (3.12b) up to terms of significant order. Comparing this solution with Eq. (3.12), one can see that for either $1+O(\alpha^{1/3})<w<\beta^{-\frac{1}{2}}$, or $0<w<1-O(\alpha^{1/3})$ the correction to the value of w at η due to the variation in μ is at most of order $O(\alpha^{2/3})$. A correction of the same order also applies to θ. Thus we may conclude that the effect of varying μ is rather unimportant in this range.

In the transonic region, we can again try to obtain the solution of Eqs. (7.3), (7.4) in the form of Eqs. (4.1)-(4.3). This substitution leads to the following equations:

$$(1-\epsilon)\frac{d^2 w^{(1)}}{d\xi^2}+2w^{(1)}\frac{dw^{(1)}}{d\xi}=(1-\beta) \tag{7.6}$$

$$(1-\epsilon)\frac{d^2 w^{(2)}}{d\xi^2}+2\frac{d}{d\xi}\left(w^{(1)}w^{(2)}\right)=\left[(\gamma-1)+\epsilon(2-\gamma)\right]\frac{d^2}{d\xi^2}\left(\frac{w^{(1)2}}{2}\right)+\frac{1+\epsilon/3}{1-\epsilon}\frac{d}{d\xi}\left(w^{(1)3}\right)$$
$$-\frac{(1+\beta)+\epsilon(1-3\beta)}{1-\epsilon}w^{(1)}+\epsilon\left[\left(\frac{dw^{(1)}}{d\xi}\right)^2+\frac{\gamma+1}{4\gamma}\frac{1}{\sigma(1+k)}\frac{d^3 w^{(1)}}{d\xi^3}\right], \tag{7.7}$$

and

$$\theta(\xi)=1-\alpha^{1/3}(\gamma-1)w^{(1)}(\xi)-\alpha^{2/3}\left(\frac{\gamma-1}{2}\right)\left[w^{(1)2}+2w^{(2)}+\frac{\epsilon}{\beta}\frac{dw^{(1)}}{d\xi}\right]+O(\alpha) \tag{7.8}$$

Putting first $\epsilon=0$ in these equations and then comparring them with the corresponding equations (4.5)-(4.7) for the case of constant μ, we see that the effect of varying μ is the introduction of an extra term to the second order equation, namely, the first term on the right hand side of Eq. (7.7). This again implies that to account for the effect of variation in μ, the values of w and θ should be subjected to a correction term of order $O(\alpha^{2/3})$. It can also be shown that in the inner supersonic region, its effect is also of $O(\alpha^{2/3})$. For the case where the dependence of μ on T is different from Eq. (7.1) (for instance $\mu = c\,T^{n}$, $n > 0$), the effect should qualitatively remain the same.

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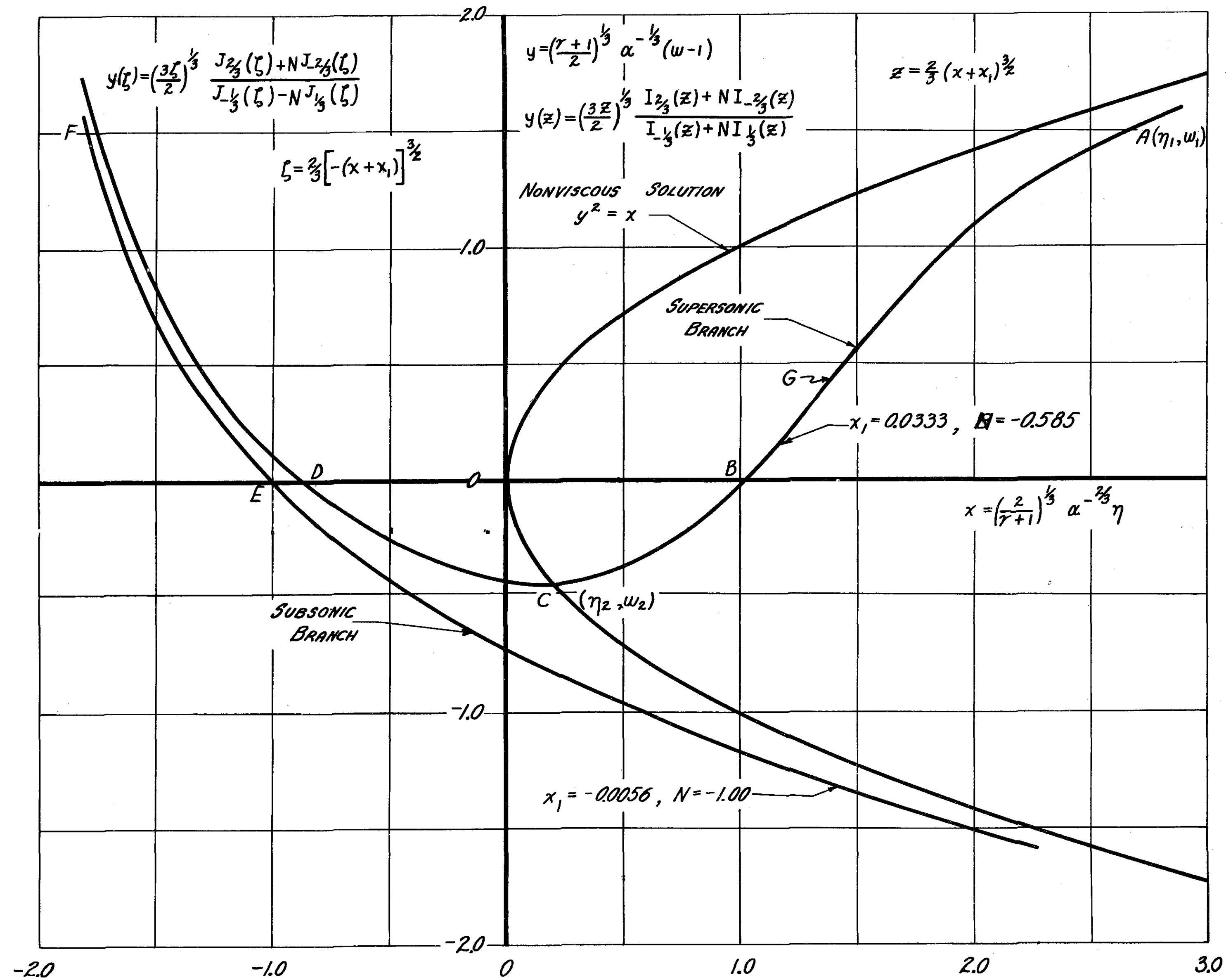

Fig. 6 - The flow velocity in the transonic region.

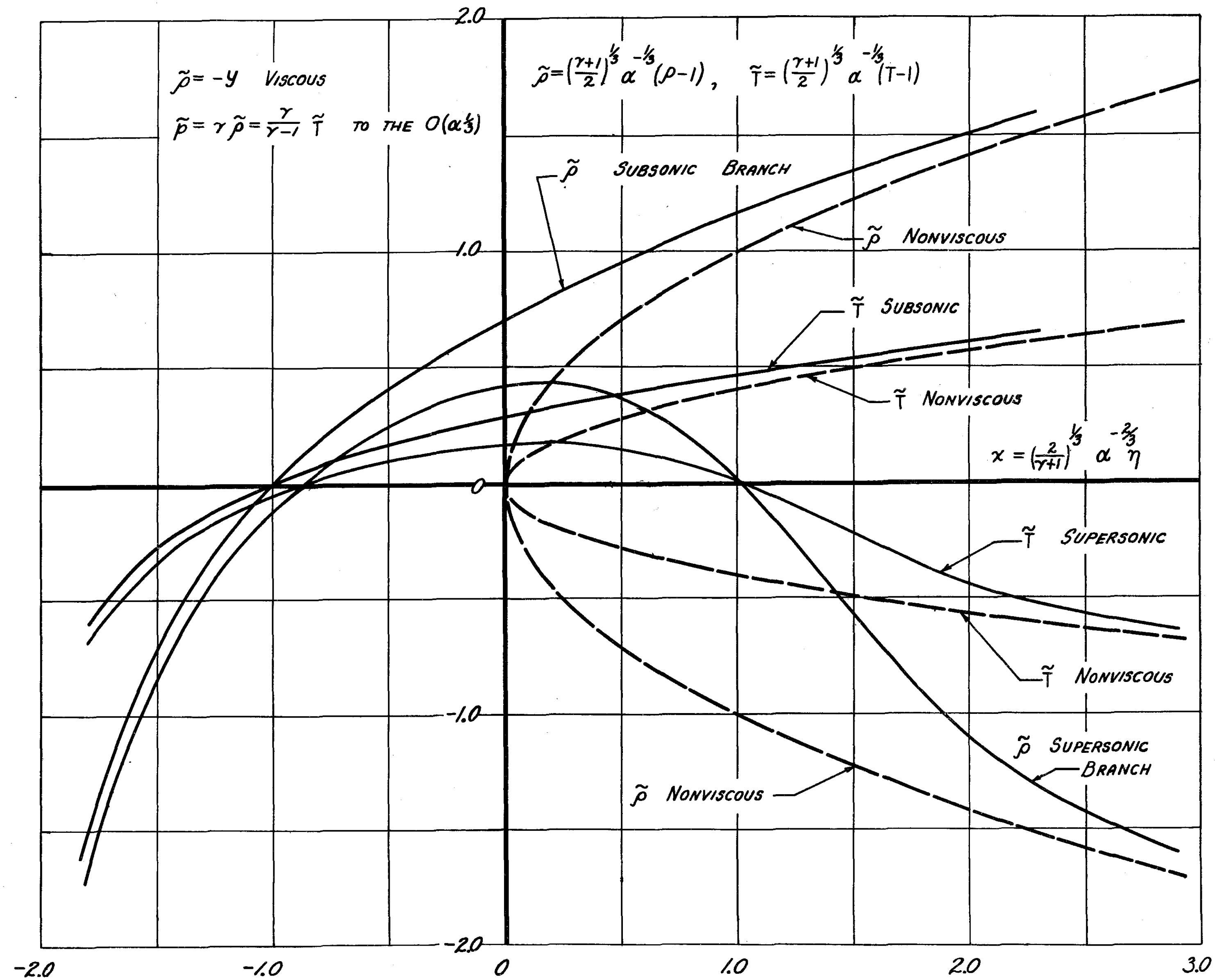

Fig. 7 - The thermodynamic variables in the transonic region.

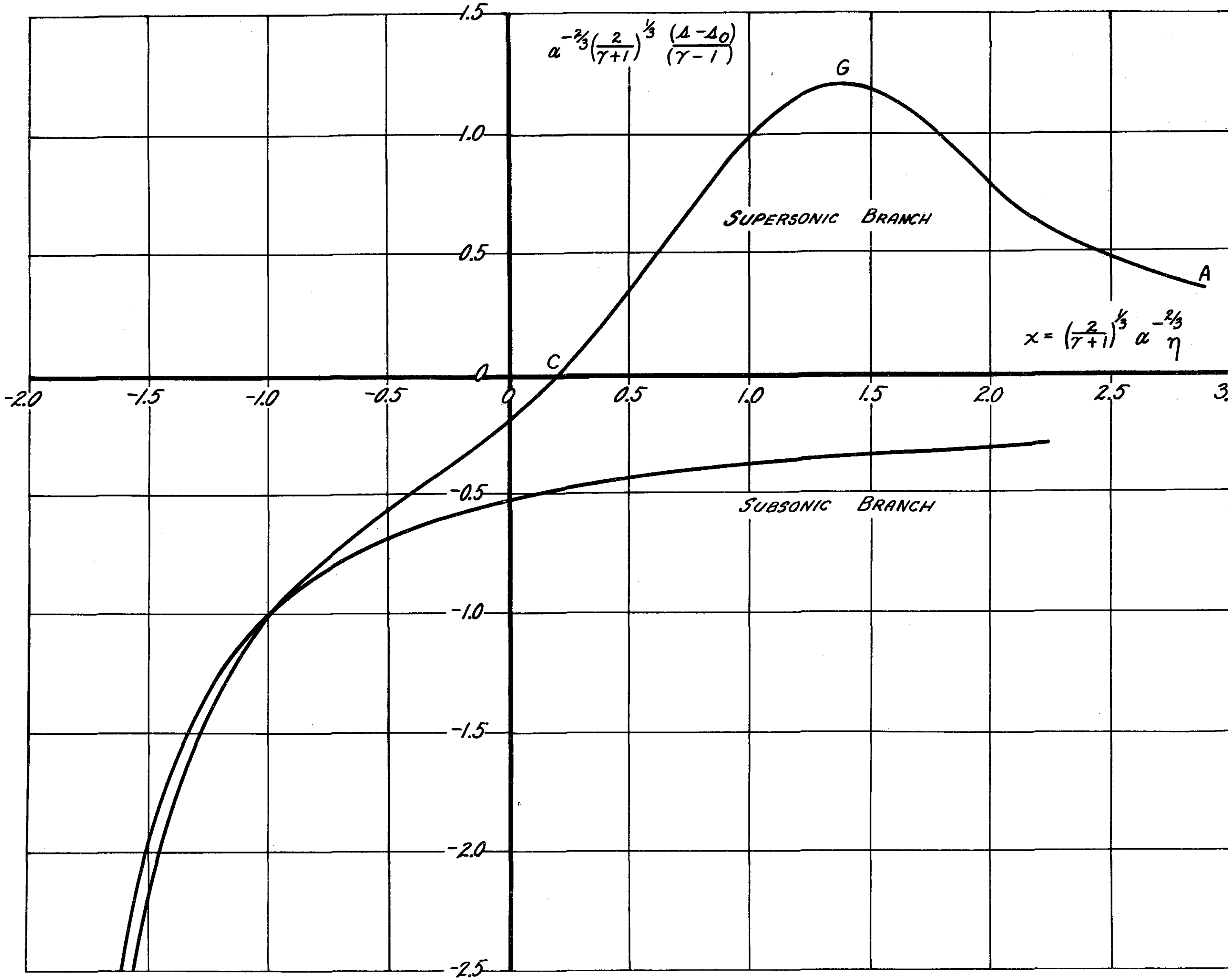

Fig. 8 - The entropy variation across the transonic region.